Extending the Tractability Border for Closest Leaf Powers

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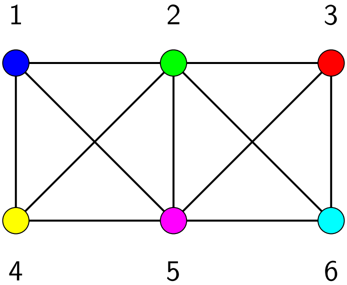
Structure of the Talk

- **Introduction and Motivation**
- Some Basic Concepts and Ideas
- A Fixed-Parameter Algorithm for Closest 4-Leaf Power

Leaf Roots and Leaf Powers



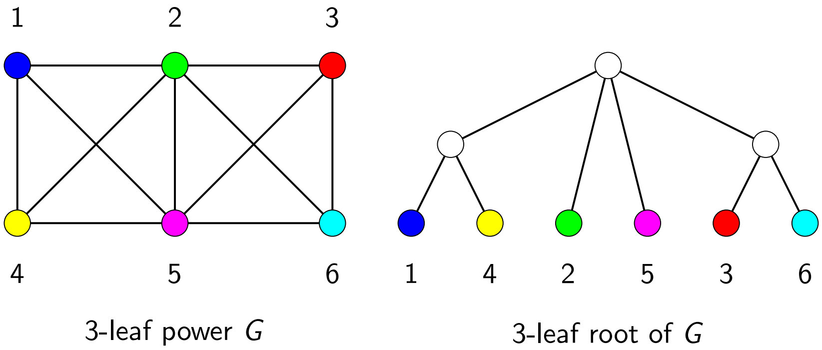
Definition

A graph $G = (V, E_G)$ is a *k-leaf power* if there is a tree $T = (V \cup S, E_T)$ with leaf set V and

$$\forall u, v \in V : \ \mathrm{dist}_T \leq k \ \Leftrightarrow \ \{u, v\} \in E_G .$$

T is called a *k-leaf root* of G.

Leaf Roots and Leaf Powers



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Leaf Power Recognition / Computing Leaf Roots

k-LEAF POWER

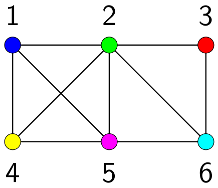
Input: A graph G.
Question: Is G a k-leaf power (has G a k-leaf root)?

Complexity of k-LEAF POWER:

- $O(|V| + |E|)$ for $k = 2$ and $k = 3$
- $O(|V|^3)$ for $k = 4$
  [N. Nishimura, P. Ragde, D. M. Thilikos, *J. Algorithms*, 2002]
- unknown for $k \geq 5$

A Graph Modification Problem

What to do if a given graph has no k-tree root?



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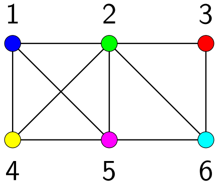

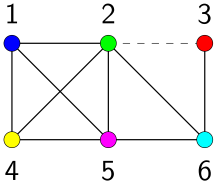

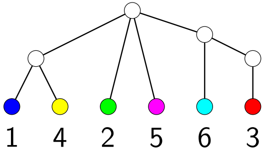

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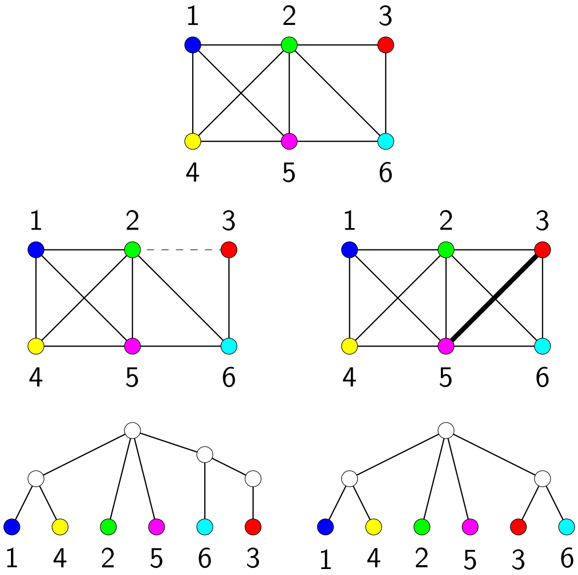

A Graph Modification Problem

CLOSEST k-LEAF POWER (CLPk)

Input: A graph G, a natural number ℓ.
Question: Is there a k-leaf power G' such that G' and G differ by at most ℓ edges?

Complexity of CLOSEST k-LEAF POWER:

- NP-complete for $k = 2$
  [M. Křivánek and J. Morávek, *Acta Informatica*, 1986]
- NP-complete for every $k \geq 3$
  [M. Dom, J. Guo, F. Hüffner, R. Niedermeier, *15th ISAAC*, 2004]
- No approximation is known for $k \geq 3$.

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Fixed-Parameter Tractability

Definition of Fixed-Parameter Tractability (FPT):

- Problem instance (G, ℓ)
- Runtime $f(\ell) \cdot |G|^{O(1)}$

- CLP2 and CLP3 are fixed-parameter tractable with respect to the parameter ℓ.
  [J. Gramm, J. Guo, F. Hüffner, R. Niedermeier, *Theory of Computing Systems*, 2005]
  [M. Dom, J. Guo, F. Hüffner, R. Niedermeier, *15th ISAAC*, 2004]

- **Now we will show fixed-parameter tractability with respect to the number of editing operations ℓ for CLP4.**

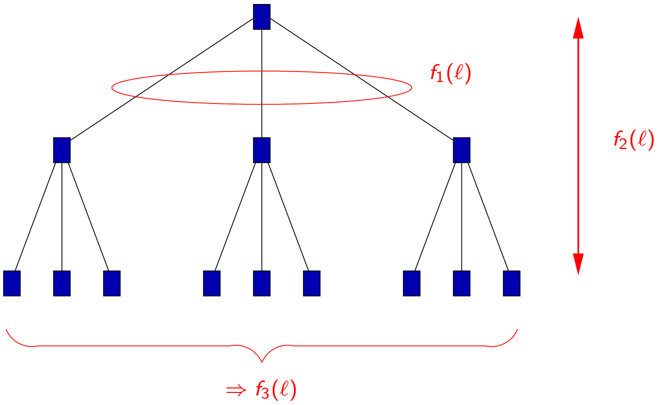
Forbidden Subgraph Characterization

We will make use of a forbidden subgraph characterization:

- Graph property Π ("is k-leaf power")
- Set $\mathcal{F}$ of forbidden subgraphs
- $G \in \Pi$
  $\Leftrightarrow$
  G does not contain any of the subgraphs in $\mathcal{F}$ as induced subgraph

Search Tree Algorithms

Search tree algorithm to transform a graph G into a Π-graph:

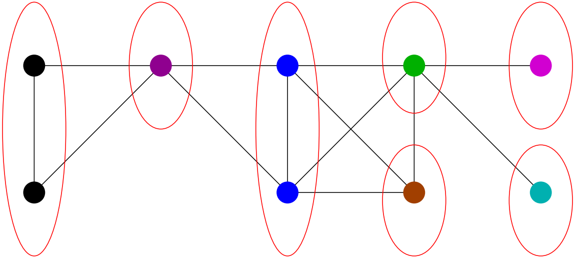


$\mathcal{F}$ finite $\Rightarrow$ fixed-parameter algorithm (running time $f_3(\ell) \cdot n^{O(1)}$).

Critical Cliques

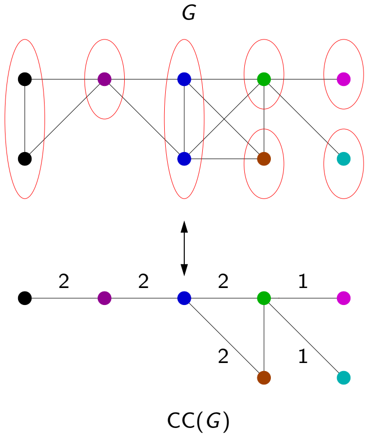
A *critical clique* of a graph G is a clique K where the vertices of K all have the same set of neighbors in $G \setminus K$, and K is maximal under this property.

[G.-H. Lin, P. E. Kearney, T. Jiang, *11th ISAAC*, 2000]



The Critical Clique Graph

Given a graph G. The *critical clique graph* $CC(G)$ has the critical cliques of G as nodes, and two nodes are connected iff the corresponding critical cliques form a larger clique in G.



Simplification of the Graph

Operate on CC(G) instead of G:

Lemma

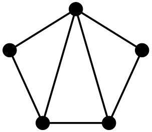
There is always an optimal solution that does not delete any edges within a critical clique and that deletes or inserts either all or no edges between two critical cliques.

Forbidden Subgraphs for Leaf Powers

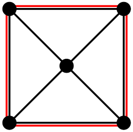
Lemma

If a graph G has a k-leaf root for any k, then G is chordal.

(A graph G is chordal, iff it contains no induced cycle of length at least four.)



chordal



not chordal

Moreover, if a graph G is chordal, then its critical clique graph $\mathrm{CC}(G)$ is chordal.

However:

- Induced cycles are not the only forbidden subgraphs.
- The set of forbidden subgraphs $C_4, C_5, C_6, \ldots$ is not finite.

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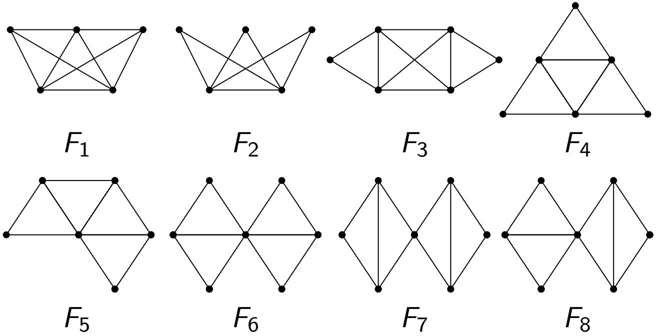
Forbidden Subgraphs for 4-Leaf Powers

A graph G is a 4-leaf power iff its critical clique graph $\mathrm{CC}(G)$ is chordal and contains no graph from the set $\mathcal{F} := \{F_1, \ldots, F_8\}$ as an induced subgraph.



Algorithm for Closest 4-Leaf Power

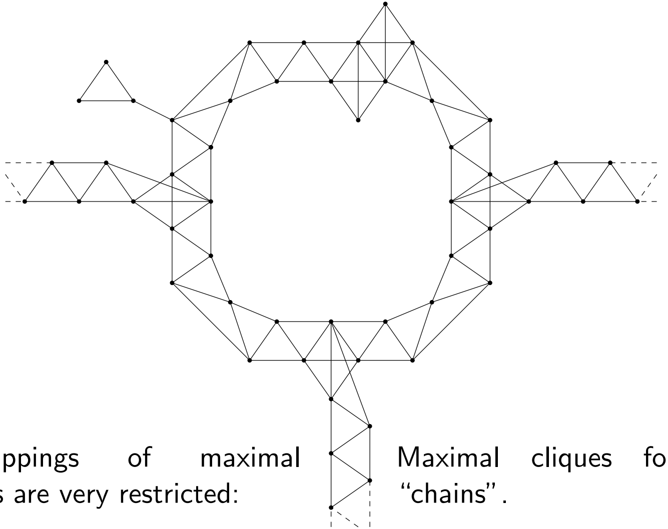
1. Destroy all forbidden subgraphs $F_1, \ldots, F_8$ in CC(G) (⇒ fixed-parameter search tree algorithm).

2. While there is a "small" induced cycle (length $\leq \ell + 3$) in CC(G):
   - delete an edge or
   - insert an edge

   (⇒ fixed-parameter search tree algorithm)

3. **But:** How to destroy "long" induced cycles?

A Closer Look at $\mathcal{F}$-Free Critical Clique Graphs



Overlappings of maximal cliques are very restricted:

Maximal cliques form "chains".

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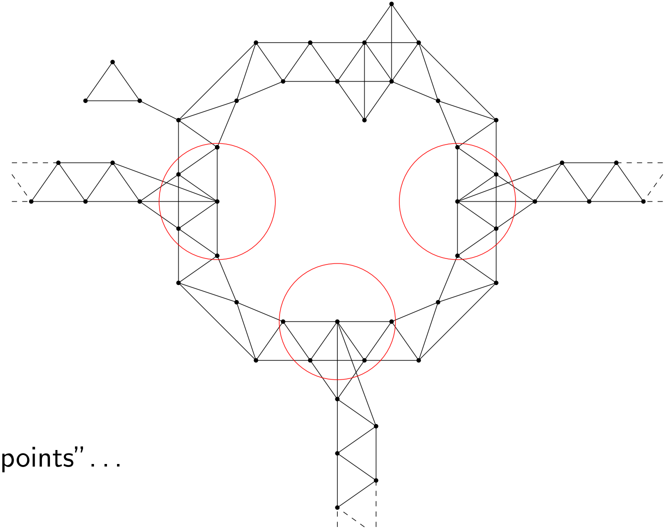
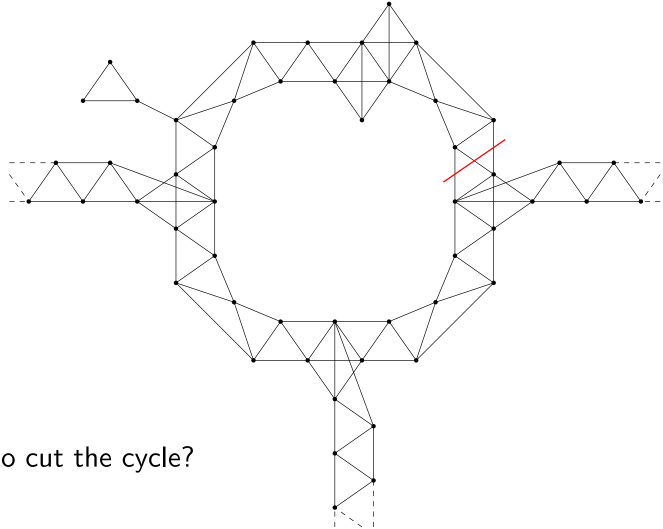


"Key points"...

A Closer Look at $\mathcal{F}$-Free Critical Clique Graphs



How to cut the cycle?

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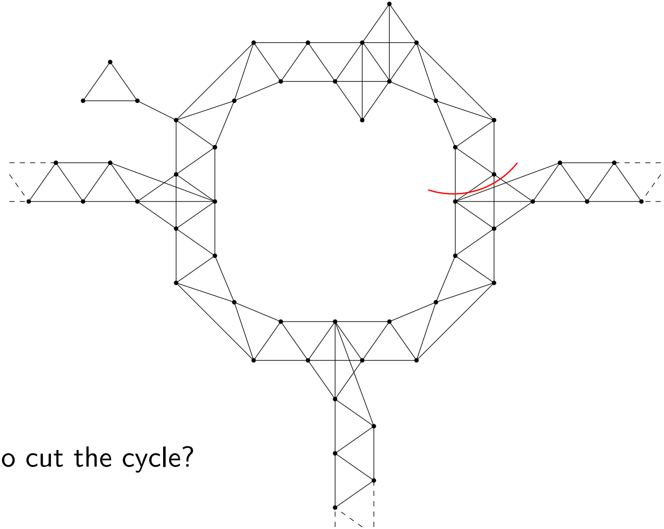

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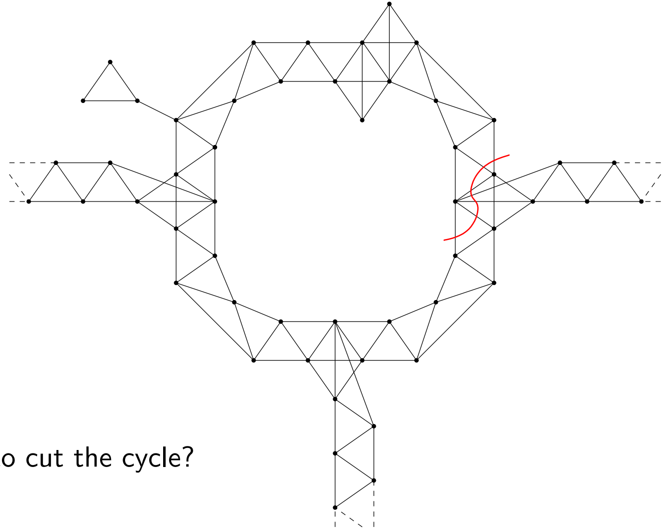

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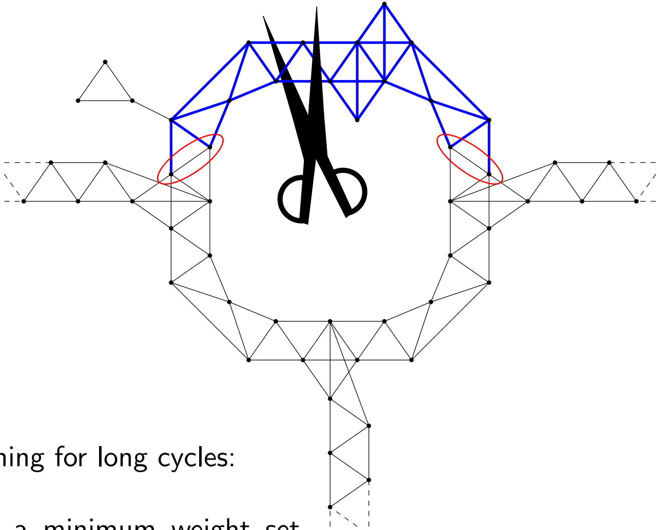

Branching for long cycles:

Delete a minimum weight set of edges between the "key points"...

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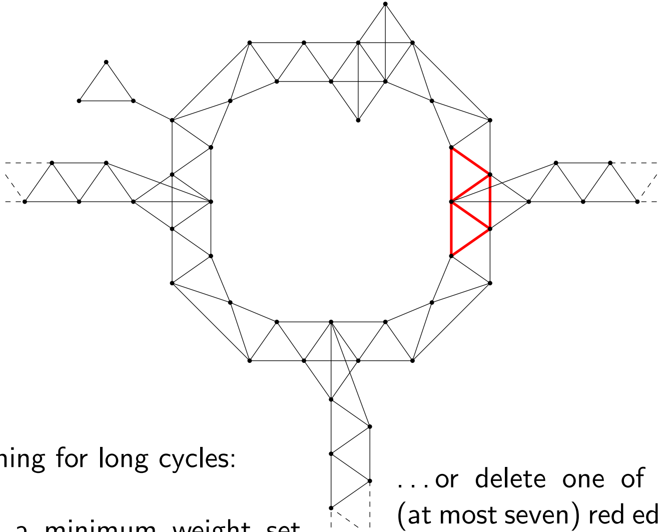


Branching for long cycles:

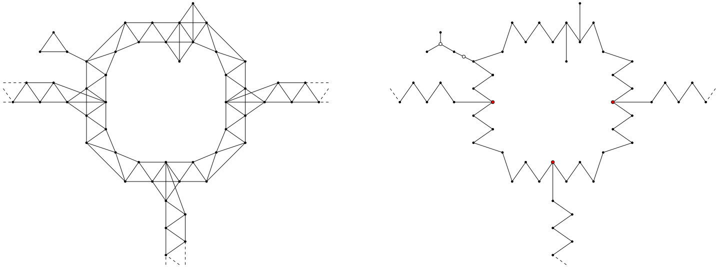
Delete a minimum weight set of edges between the "key points"...

...or delete one of the (at most seven) red edges at a "key point".

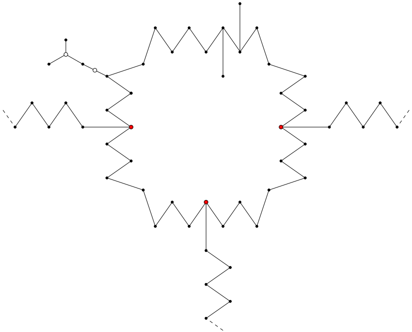
Bounding the Number of "Key Points" (1)

Build a "Pseudo Steiner Root" S for the $\mathcal{F}$-free critical clique graph $\mathrm{CC}(G)$:

- $dist_{\mathrm{CC}(G)}(U, V) = 1 \Leftrightarrow dist_S(U, V) \leq 2$.
- The nodes of a cycle in S induce at least one cycle in $\mathrm{CC}(G)$.
- Each "key point" in $\mathrm{CC}(G)$ corresponds to a node of degree at least 3 in S.



Bounding the Number of "Key Points" (2)



Theorem

Every graph with minimum vertex degree at least 3 contains a cycle of length at most $2 \log n + 1$.

[Erdős and Pósa]

Running Time of the Algorithm

Branching of the search tree algorithm:

- $2 \log n + 1$ "key points"
- Eight possibilities for each "key point"

Running time: $(48 \cdot O(\log n) + 24)^{\ell} \cdot n^{O(1)} = c^{\ell} \cdot (\ell \log \ell)^{\ell} \cdot n^{O(1)}$

Theorem

CLOSEST 4-LEAF POWER *is fixed-parameter tractable with respect to the parameter ℓ (number of modifications).*

Open Questions

- Generalization to Closest k-Leaf Power for $k > 4$:
  - Can graphs that have a k-leaf root be recognized in polynomial time?
  - Is there a useful characterization by a small set of forbidden subgraphs?
- Extension to the closely related problem Closest Phylogenetic k-th Power?
- How small can the combinatorial explosion for CLP3, CLP4 and their variants in the parameter ℓ (number of modifications) be made?