Parameterized Complexity of Stabbing Rectangles and Squares in the Plane

Michael Dom[1]

Michael R. Fellows[2,3] Frances A. Rosamond[2,3]

[1]Institut für Informatik, Friedrich-Schiller-Universität Jena, Germany

[2]PC Research Unit, University of Newcastle, Australia

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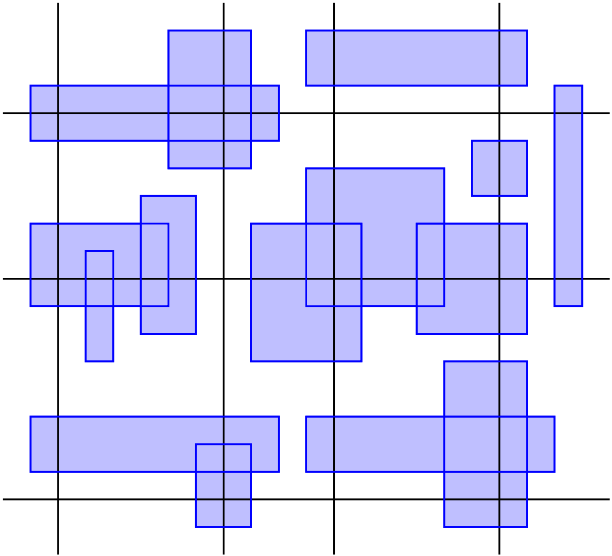
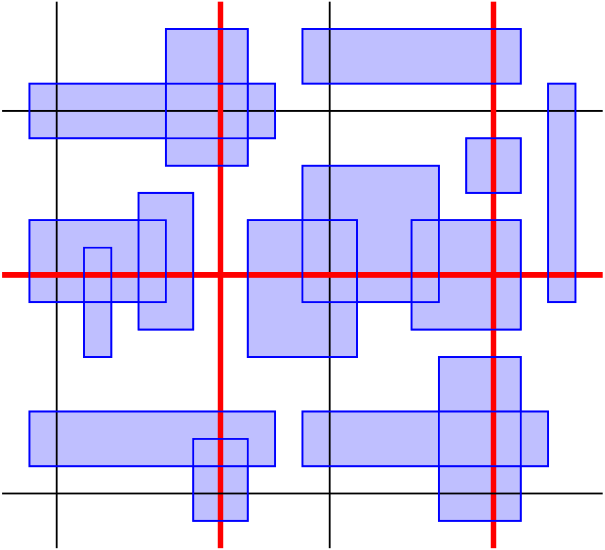
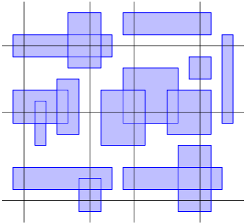
Rectangle Stabbing



Rectangle Stabbing



Rectangle Stabbing

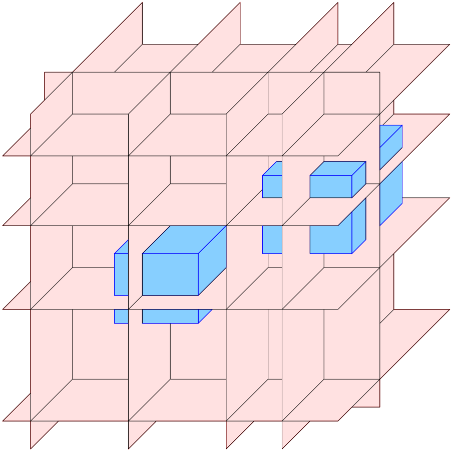


Rectangle Stabbing

Input: A set R of axis-parallel rectangles, a set L of axis-parallel lines, a positive integer k.

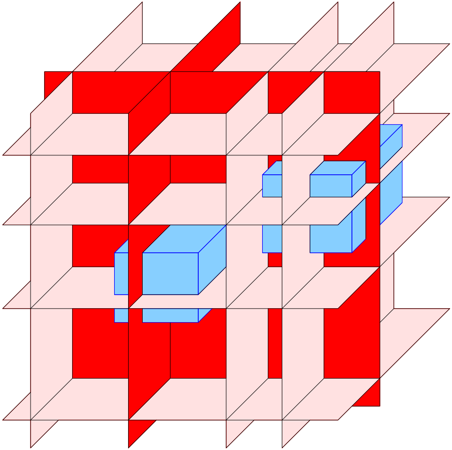
Question: Exists $L' \subseteq L$ with $|L'| \leq k$ such that every rectangle from R is intersected by at least one line from L'?

3-Dimensional Rectangle Stabbing



Generalization to d dimensions:
d-Dimensional Rectangle Stabbing.

3-Dimensional Rectangle Stabbing



Generalization to d dimensions:
d-Dimensional Rectangle Stabbing.

Set Cover

1	2	3	4	5	6
	1			1	1
1			1	1	
	1	1			
1			1		1
		1	1	1	1

Set Cover
Input: A binary matrix M, a positive integer k.
Question: Is there a set of at most k columns that hits a 1 in every row?

Set Cover

1	2	3	4	5	6
	1			1	1
1			1	1	
	1	1			
1			1		1
		1	1	1	1

Set Cover
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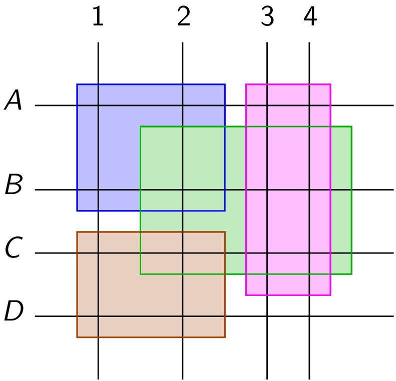
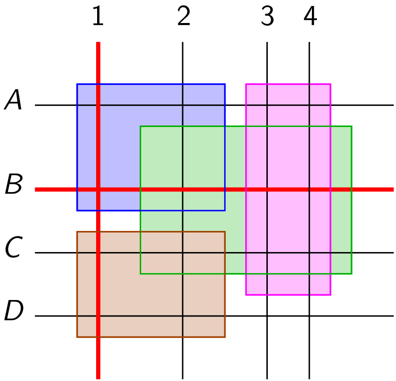
Rectangle Stabbing and Set Cover



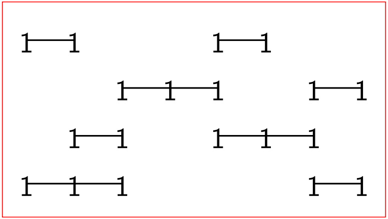
1	2	3	4	A	B	C	D
1	1			1	1		
	1	1	1		1	1	
		1	1	1	1	1	
1	1					1	1

Rectangle Stabbing and Set Cover

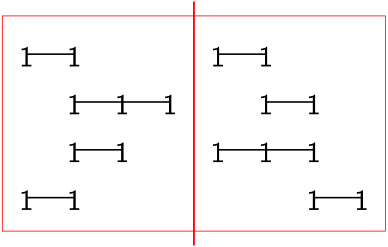


1	2	3	4	A	B	C	D
1	1			1	1		
	1	1	1		1	1	
		1	1	1	1	1	
1	1					1	1

Variants of the Consecutive-Ones Property

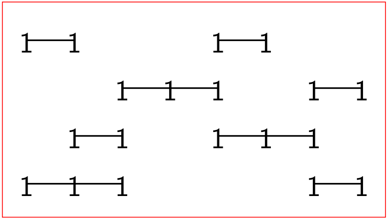


2-C1P:
two blocks per row

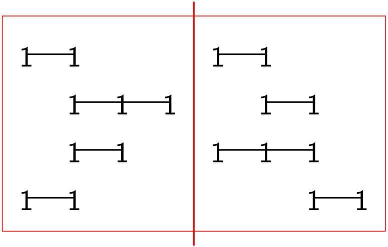


2-SC1P:
dividing line, one block per section

Restricted Variants of Set Cover



2-C1P



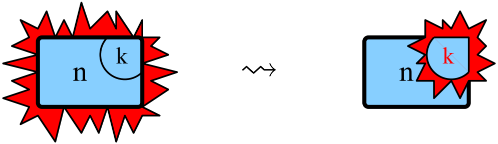
2-SC1P

2-SC1P-Set Cover and Rectangle Stabbing are equivalent.

Both problems are NP-complete.
[Gaur et al., *J. Algorithms, '02*, Mecke et al., *ATMOS '05*, Dom and Sikdar, *FAW '08*]

Parameterized Complexity

- Main idea: Measure complexity not only in input size, but also in an additional "parameter" k.



- Problem is *fixed-parameter tractable (FPT)* with respect to a parameter k

  $\Leftrightarrow$

  problem is solvable in $f(k) \cdot n^{O(1)}$ time.

  Example: $O(2^k \cdot n^2)$
  Not FPT: $O(n^k)$

- W[1]-hardness: concept for parameterized intractability

Known Results

- Factor-$d2^{d-1}$ approximation for d-Dimensional Rectangle Stabbing when all hyperrectangles are identical
  [Hassin and Megiddo, *Discrete Appl. Math., '91*]
- Factor-d approximation for d-Dimensional Rectangle Stabbing
  [Gaur et al., *J. Algorithms, '02*]
- Factor-d approximation for d-C1P-Set Cover
  [Mecke et al., *ATMOS '05*]
- Approximation algorithms for 2-Dimensional Rectangle Stabbing when every rectangle has height or width one
  [Hassin and Megiddo, *Discrete Appl. Math., '91*,
  Kovaleva and Spieksma, *ISAAC '01*, *SIAM J. Discrete Math., '06*]
- 3-Dimensional Rectangle Stabbing is W[1]-hard
  [Dom and Sikdar, *FAW '08*]
- Special cases of Rectangle Stabbing are in FPT
  [Dom and Sikdar, *FAW '08*]
- Open: Parameterized complexity of Rectangle Stabbing

Our Main Results

- Rectangle Stabbing is W[1]-hard.
- Rectangle Stabbing is W[1]-hard
  if all rectangles are squares of the same size.
- Rectangle Stabbing is in FPT
  if all rectangles are *nonoverlapping* squares of the same size.

Our Main Results

- **Rectangle Stabbing is** W[1]-**hard.**
- Rectangle Stabbing is W[1]-hard
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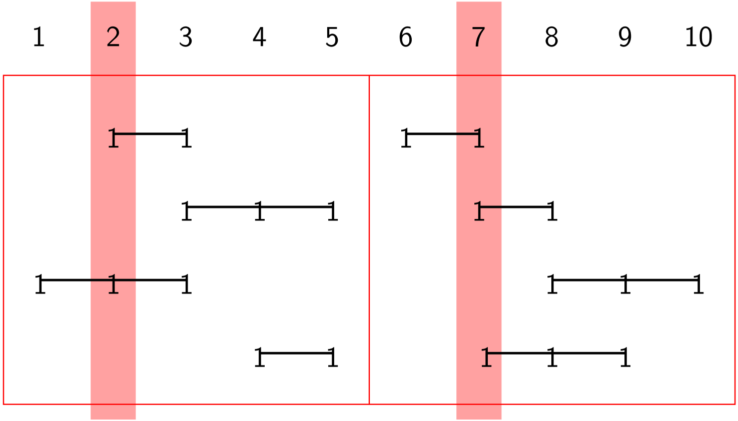
2-SC1P-Set Cover is W[1]-hard



2-SC1P-Set Cover

Input: A binary matrix M with 2-SC1P, a positive integer k.

Question: Is there a set of at most k columns that hits a 1 in every row?

2-SC1P-Set Cover is W[1]-hard

Parameterized reduction:

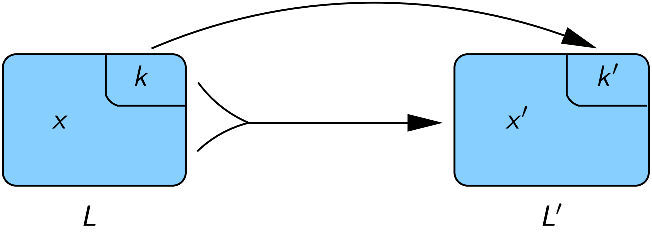
- Same basic idea as polynomial-time reduction:
  Reduce from a hard problem.

$$(x, k) \rightsquigarrow (x', k')$$

- New parameter must depend only on the old parameter:

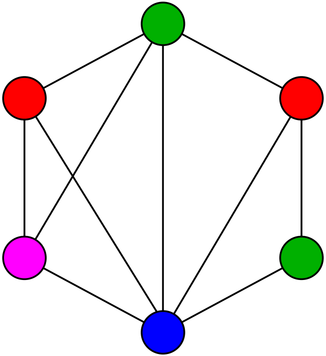
$$k' = f(k)$$

- (Reduction may cost $g(k) \cdot n^{O(1)}$ time.)



2-SC1P-Set Cover is W[1]-hard

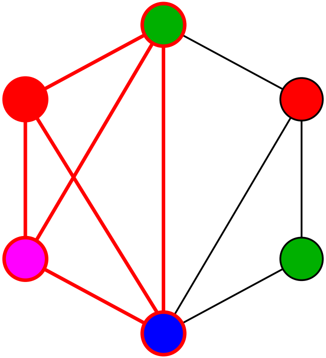
Reduction from the W[1]-hard problem Multicolored Clique.



Multicolored Clique
Input: A positive integer k and a k-colored undirected graph.
Question: Is there a clique of size k?

2-SC1P-Set Cover is W[1]-hard

Reduction from the W[1]-hard problem Multicolored Clique.

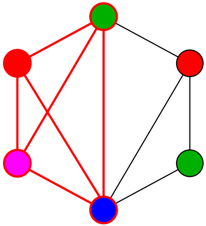


Multicolored Clique
Input: A positive integer k and a k-colored undirected graph.
Question: Is there a clique of size k?

2-SC1P-Set Cover is W[1]-hard

Reformulation of Multicolored Clique:

[Fellows et al., manuscript, 2008]



Question: Is there a set E' of $\binom{k}{2}$ edges and a set V' of k vertices such that

- E' contains an edge of every "edge color",
- V' contains a vertex of every color, and
- $\{v, w\} \in E' \rightarrow v, w \in V'$?

2-SC1P-Set Cover is W[1]-hard

Question: Is there a set E' of $\binom{k}{2}$ edges and a set V' of k vertices such that

- E' contains an edge of every "edge color",
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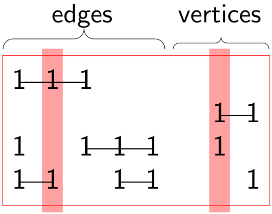
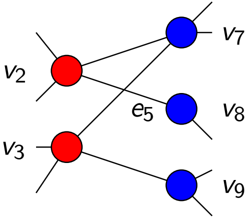
Approach for the reduction to 2-SC1P-Set Cover:



- one column for every edge and every vertex
- number of columns to select: $\binom{k}{2} + k$
- rows to enforce the three constraints

2-SC1P-Set Cover is W[1]-hard

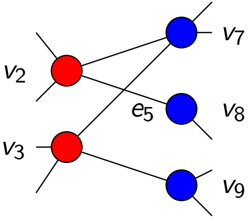


- E' contains an edge of every "edge color",
- V' contains a vertex of every color, and
- $\{v, w\} \in E' \rightarrow v, w \in V'$?

...	{red, blue}				...	...	red		...	blue			...
...	e_4	e_5	e_6	e_7	...	...	v_2	v_3	...	v_7	v_8	v_9	...
	1	1	1	1									
							1	1					
										1	1	1	
	1		1	1			1						
	1		1	1							1		

Number of columns to select: $\binom{k}{2} + k$.

2-SC1P-Set Cover is W[1]-hard



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...	e_4	e_5	e_6	e_7	...	...	v_2	v_3	...	v_7	v_8	v_9	...
	1	1	1	1									
							1	1					
										1	1	1	
	1		1	1			1						
	1		1	1							1		

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2-SC1P-Set Cover is W[1]-hard

...	{red, blue}				...			...	...	red		...	blue			...
...	e_4	e_5	e_6	e_7	...			...	...	v_2	v_3	...	v_7	v_8	v_9	...
	1	1	1	1												
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					1	1										

2-SC1P-Set Cover is W[1]-hard

...	{red, blue}				{red, blue}				...	...	red		...	blue			...	...	{red, blue}				...
...	e_4	e_5	e_6	e_7	e_4	e_5	e_6	e_7	...	...	v_2	v_3	...	v_7	v_8	v_9	...	...	e_4	e_5	e_6	e_7	...
	1	1	1	1																			
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			1	1	1										1								

#col's to select: $3 \cdot \binom{k}{2} + k$

2-SC1P-Set Cover is W[1]-hard

...	{red, blue}				{red, blue}				...	...	red		...	blue			...	...	{red, blue}				...
...	e_4	e_5	e_6	e_7	e_4	e_5	e_6	e_7	...	...	v_2	v_3	...	v_7	v_8	v_9	...	...	e_4	e_5	e_6	e_7	...
	1	1	1	1																			
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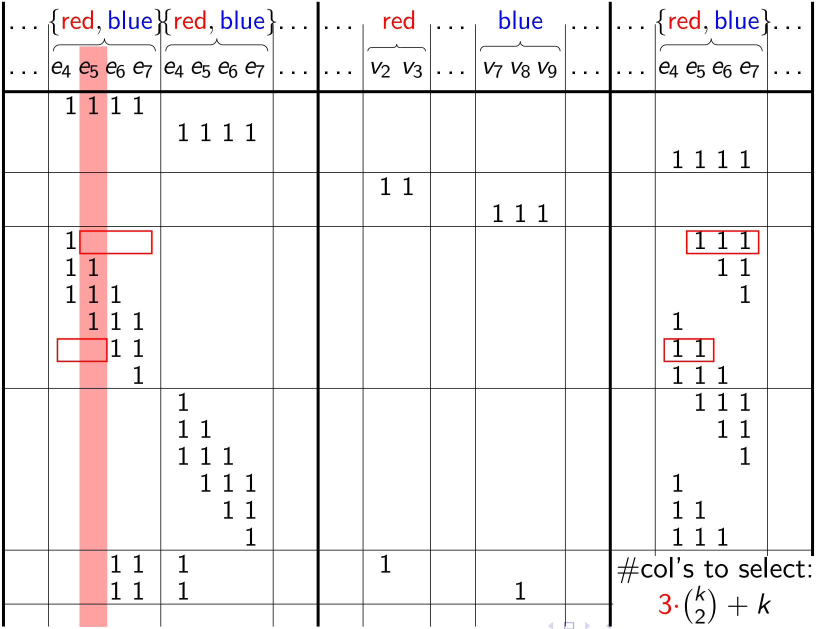
2-SC1P-Set Cover is W[1]-hard



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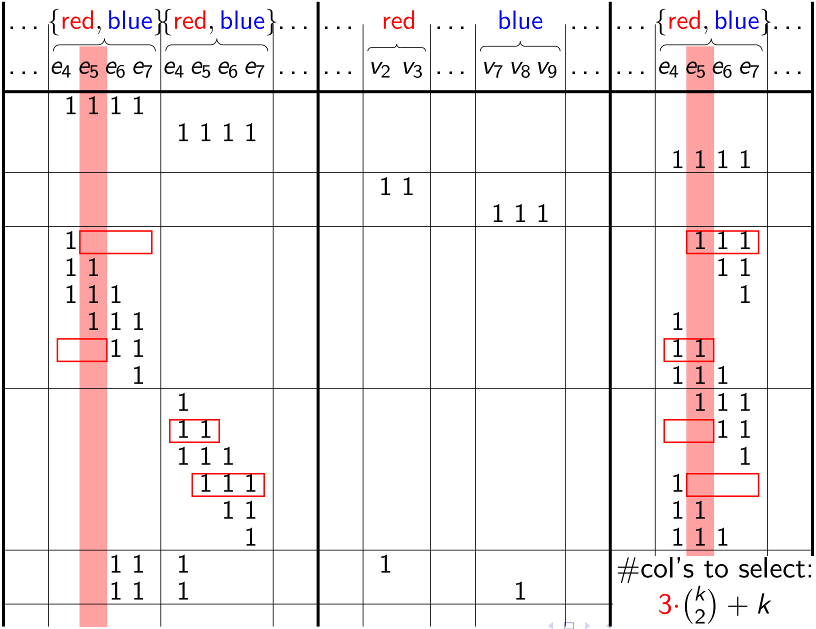
2-SC1P-Set Cover is W[1]-hard



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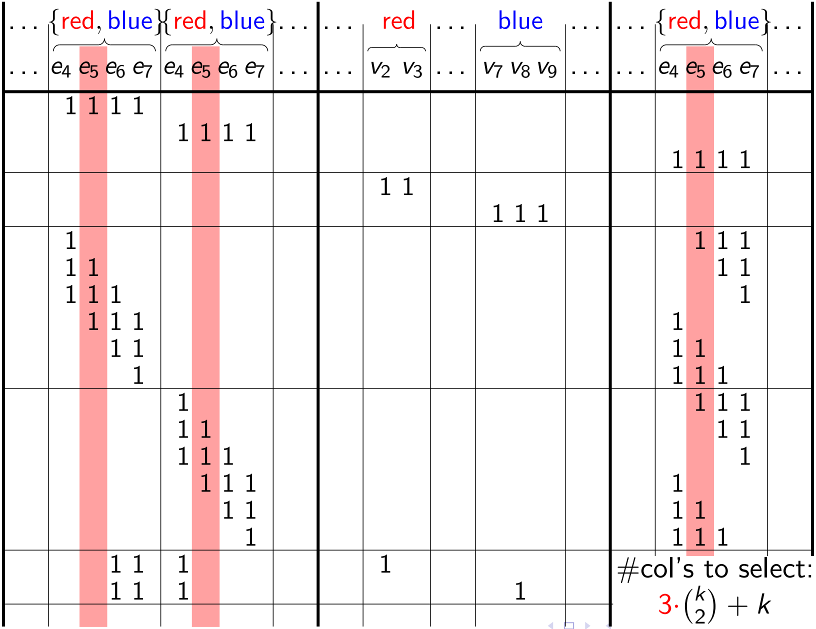

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2-SC1P-Set Cover is W[1]-hard

...	{red, blue}				{red, blue}				...	...	red		...	blue			...	...	{red, blue}				...
...	e_4	e_5	e_6	e_7	e_4	e_5	e_6	e_7	...	...	v_2	v_3	...	v_7	v_8	v_9	...	...	e_4	e_5	e_6	e_7	...
	1	1	1	1																			
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			1	1	1						1												
			1	1	1										1								

2-SC1P-Set Cover is W[1]-hard

...	{red, blue}				{red, blue}				...	...	red		...	blue			...	...	{red, blue}				...
...	e_4	e_5	e_6	e_7	e_4	e_5	e_6	e_7	...	...	v_2	v_3	...	v_7	v_8	v_9	...	...	e_4	e_5	e_6	e_7	...
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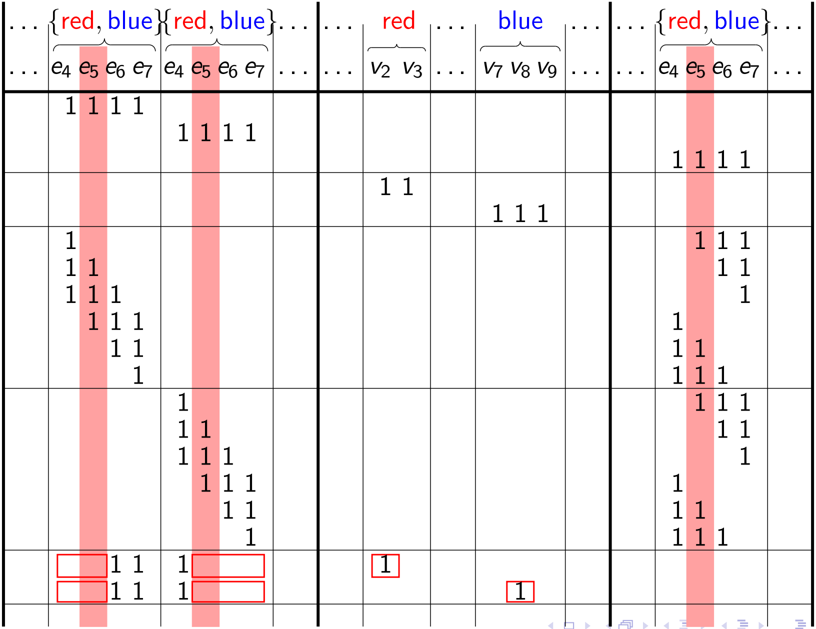
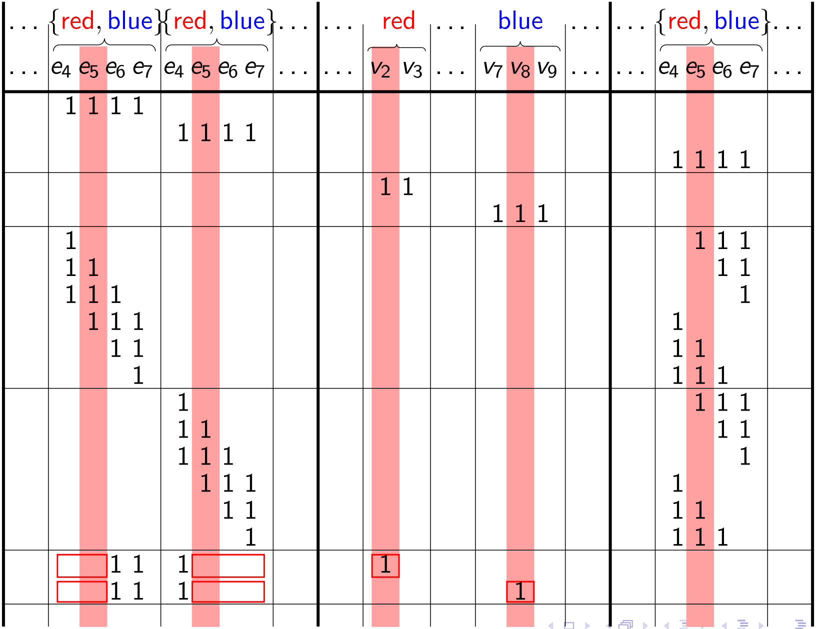
2-SC1P-Set Cover is W[1]-hard



2-SC1P-Set Cover is W[1]-hard



2-SC1P-Set Cover is W[1]-hard

...	{red, blue}				{red, blue}				...	...	red		...	blue			...	...	{red, blue}				...
...	e_4	e_5	e_6	e_7	e_4	e_5	e_6	e_7	...	...	v_2	v_3	...	v_7	v_8	v_9	...	...	e_4	e_5	e_6	e_7	...
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2-SC1P-Set Cover is W[1]-hard

Theorem

2-SC1P-Set Cover, 2-C1P-Set Cover, and Rectangle Stabbing are W[1]*-hard with respect to the parameter k.*

With a similar reduction:

Theorem

Rectangle Stabbing is W[1]*-hard with respect to the parameter k if all rectangles are squares of the same size.*

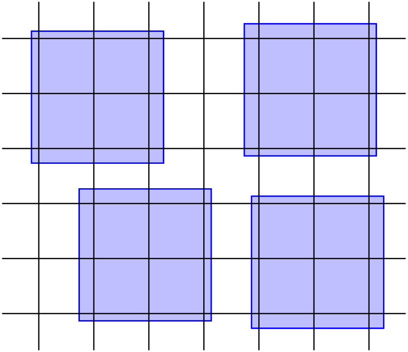
Our Main Results

- Rectangle Stabbing is W[1]-hard.
- Rectangle Stabbing is W[1]-hard
  if all rectangles are squares of the same size.
- **Rectangle Stabbing is in** FPT
  if all rectangles are nonoverlapping squares of the same size.

Stabbing Nonoverlapping Identical Squares

Restrictions:

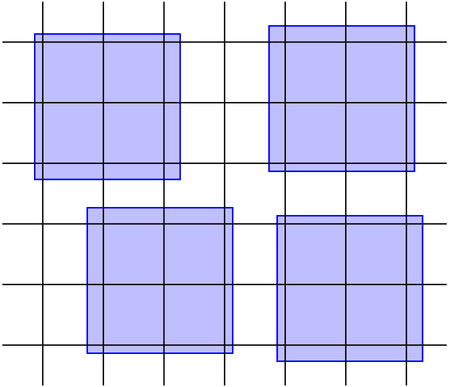
- There is a number b such that each rectangle is intersected by exactly b vertical lines and exactly b horizontal lines.
- No two rectangles are intersected by a common vertical line *and* a common horizontal line.



NP-hard if $b \geq 2$.

Stabbing Nonoverlapping Identical Squares

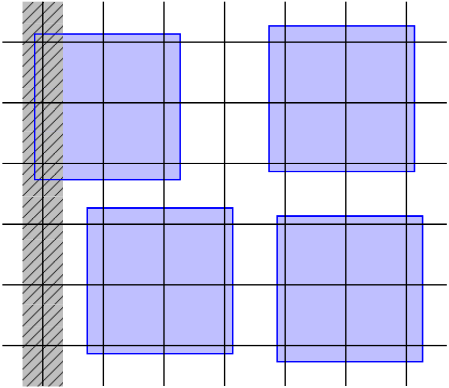
Use data reduction rules:



1. Delete "dominated" lines.

Stabbing Nonoverlapping Identical Squares

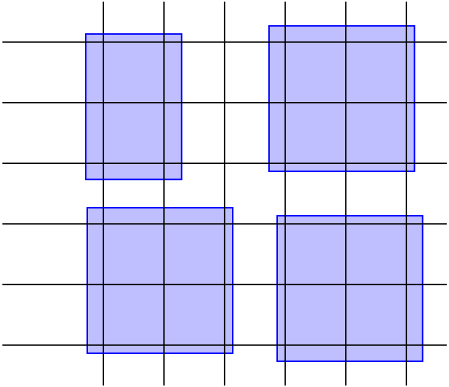
Use data reduction rules:



1. Delete "dominated" lines.

Stabbing Nonoverlapping Identical Squares

Use data reduction rules:



1. Delete "dominated" lines.

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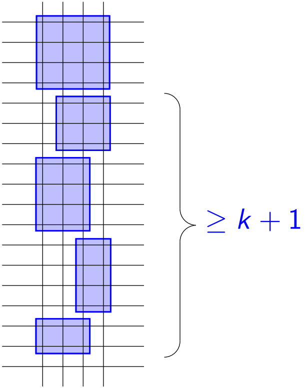

1. Delete "dominated" lines.
2. Delete "unnecessary" rectangles.

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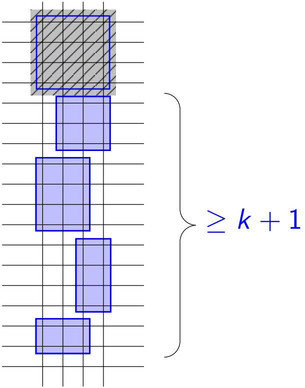

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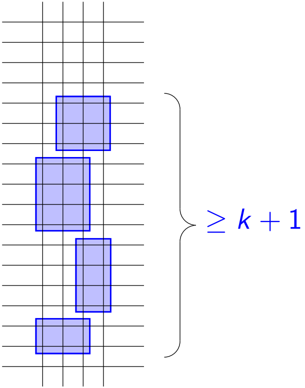

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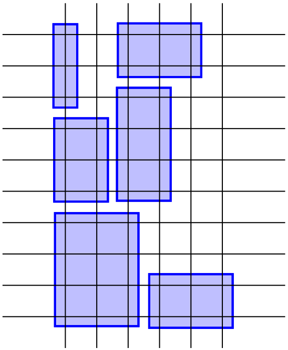
Properties of reduced problem instances:



1. At each vertical line, there “ends” at least one rectangle.

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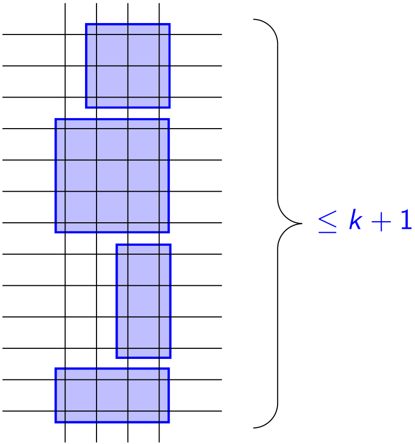

1. At each vertical line, there "ends" at least one rectangle.
2. At each vertical line, there end at most $k + 1$ rectangles.

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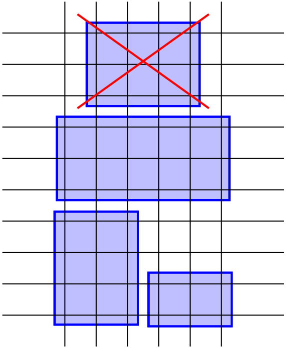

1. At each vertical line, there "ends" at least one rectangle.
2. At each vertical line, there end at most $k + 1$ rectangles.
3. No rectangle starts later *and* ends earlier than another one.

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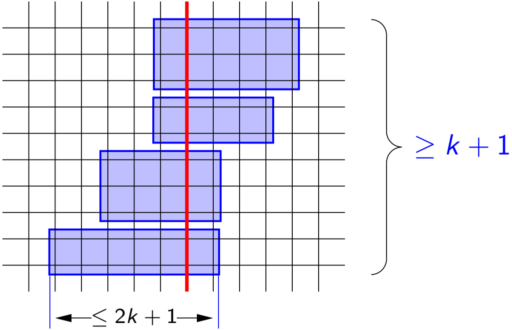
Properties of reduced problem instances:



Lemma

Either each rectangle is intersected by at most $2k+1$ vertical lines or
there is a vertical line that intersects more than k rectangles, such that each of these rectangles is intersected by at most $2k+1$ vertical lines.

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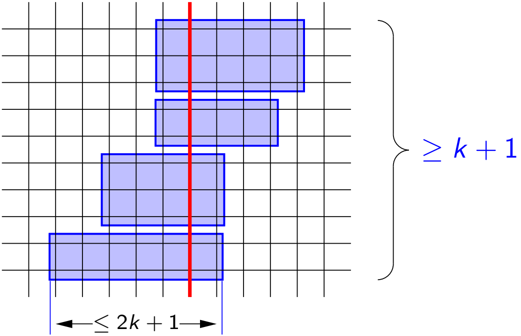

Theorem

Rectangle Stabbing can be solved in $(4k+1)^k \cdot n^{O(1)}$ *time if all rectangles are nonoverlapping squares of the same size.*

Conclusion

We know: Rectangle Stabbing is. . .

- . . . W[1]-hard for the parameter k
  if all rectangles are squares of the same size.

- . . . in FPT for the parameter k
  if all rectangles are *nonoverlapping* squares of the same size.

Open:

- Is Rectangle Stabbing in FPT for the parameter k
  if all rectangles are *nonoverlapping*?

Conclusion

We know:

- Rectangle Stabbing is in FPT for the parameter k
  if all rectangles are *nonoverlapping* squares of the same size.

Open:

- Is there a polynomial-size kernel?

Conclusion

We know (not part of this talk):

- d-C1P-Set Cover and d-Dimensional Rectangle Stabbing with *constant* d are in W[1] for the parameter k.

Open:

- Are these problems in W[1] for the parameter d, k, or are they W[2]-hard?

Conclusion

We know (not part of this talk):

- d-C1P-Set Cover and d-Dimensional Rectangle Stabbing with *constant* d are in W[1] for the parameter k.

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- Are these problems in W[1] for the parameter d, k, or are they W[2]-hard?

Thank you.