Approximation and Fixed-Parameter Algorithms for Consecutive Ones Submatrix Problems

Michael Dom, Jiong Guo, and Rolf Niedermeier

Institut für Informatik, Friedrich-Schiller-Universität Jena, Germany

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Consecutive Ones Property (C1P)

$$
\begin{array}{ccccccccccc}
1 & 1 & 1 & 1 & & & & & & & \\
1 & 1 & 1 & 1 & 1 & & & & & & \\
 & & & & & 1 & 1 & 1 & 1 & 1 & \\
 & & & & & & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & &
\end{array}
$$

A 0/1-matrix has the C1P if its columns can be permuted such that in each row the ones form a block.

Consecutive Ones Property (C1P)

Example for a matrix having the C1P:

$$
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\boxed{\begin{array}{ccccc} 1 & 1 & & & 1 \\ 1 & & 1 & & 1 \\ 1 & & 1 & 1 & \end{array}}
\end{array}
$$

Consecutive Ones Property (C1P)

Example for a matrix having the C1P:

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \boxed{\begin{array}{ccccc} 1 & 1 & & & 1 \\ 1 & & 1 & & 1 \\ 1 & & 1 & 1 & \end{array}} \end{array}$$

$$\begin{array}{c} \begin{array}{ccccc} 2 & 5 & 1 & 3 & 4 \end{array} \\ \boxed{\begin{array}{ccccc} 1 & 1 & 1 & & \\ & 1 & 1 & 1 & \\ & & 1 & 1 & 1 \end{array}} \end{array}$$

Consecutive Ones Property (C1P)

Examples for matrices **not** having the C1P:

$$\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{array}$$

$$\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{array}$$

$$\begin{array}{ccccc} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{array}$$

$$\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}$$

$$\begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}$$

Consecutive Ones Property (C1P)

The Consecutive Ones Property...

- ...expresses "locality" of the input data.
- ...appears in many applications, e.g.
  - in railway system optimization [Ruf, Schöbel, Discrete Optimization, 2004; Mecke, Wagner, ESA '04],
  - bioinformatics [Christof, Oswald, Reinelt, IPCO '98; Lu, Hsu, J. Comp. Biology, 2003].
- ...can be recognized in polynomial time [Booth, Lueker, J. Comput. System Sci., 1976; Meidanis, Porto, Telles, Discrete Appl. Math., 1998; Habib, McConnell, Paul, Viennot, Theor. Comput. Sci., 2000, Hsu, J. Algorithms, 2002; McConnell, SODA '04].
- ...is subject of current research [Hajiaghayi, Ganjali, Inf. Process. Lett., 2002; Tan, Zhang, Algorithmica, 2007].

Problem Definition

Min-COS-C (Min-COS-R)
Given: A matrix M and a positive integer k.
Question: Can we delete at most k columns (at most k rows) such that the resulting matrix has the C1P?

Min-COS-C is NP-complete even on $(2,3)$- and $(3,2)$-matrices [Hajiaghayi, Ganjali, Inform. Process. Letters, 2002; Tan, Zhang, Algorithmica, 2007].
Min-COS-R is NP-complete even on $(3,2)$-matrices [Hajiaghayi, Ganjali, Inform. Process. Letters, 2002].

Problem Overview

(1's per col, 1's per row)	Max-COS-C	Min-COS-C
$(3, 2)$	0.5-approx[1]	
$(*, 2)$	• No const. approx.[1]	
$(*, \Delta)$	• No const. approx.[1]	
$(2, 3)$	0.8-approx[1]	
$(2, *)$	0.5-approx[1]	
$(\Delta, *)$		

[1] [Tan, Zhang, Algorithmica, 2007]

Problem Overview

(1's per col, 1's per row)	Max-COS-C	Min-COS-C
$(3, 2)$	0.5-approx[1]	
$(*, 2)$	• No const. approx.[1] • W[1]-hard	• No 2,72-approx. • Problem kernel
$(*, \Delta)$	• No const. approx.[1] • W[1]-hard	• $(\Delta + 2)$-approx. • $O((\Delta+2)^k \cdot \Delta^{O(\Delta)} \cdot \|M\|^{O(1)})$-alg.
$(2, 3)$	0.8-approx[1]	
$(2, *)$	0.5-approx[1]	• 6-approx • $O(6^k \cdot \text{pol}(\|M\|))$-alg.
$(\Delta, *)$		

[1] [Tan, Zhang, Algorithmica, 2007]

Problem Overview

(1's per col, 1's per row)	Max-COS-C	Min-COS-C
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$(\Delta, *)$		

[1] [Tan, Zhang, Algorithmica, 2007]

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

$$
\overbrace{\left.\begin{array}{|ccccc|}
\hline
\mathbf{1} & \mathbf{1} & 0 & \cdots & 0 \\
0 & \mathbf{1} & \mathbf{1} & 0\cdots & 0 \\
 & & \cdots & & \\
0 & \cdots & 0 & \mathbf{1} & \mathbf{1} \\
\hline
\mathbf{1} & 0 & \cdots & 0 & \mathbf{1} \\
\hline
\end{array}\right\}}^{p+2} p+2
\qquad
\overbrace{\left.\begin{array}{|cccccc|c|}
\hline
\mathbf{1} & \mathbf{1} & 0 & & \cdots & & 0 \\
0 & \mathbf{1} & \mathbf{1} & 0 & \cdots & & 0 \\
 & & & \cdots & & & \\
0 & \cdots & & 0 & \mathbf{1} & \mathbf{1} & 0 \\
\hline
0 & \mathbf{1} & & & \cdots & & \mathbf{1} \\
\mathbf{1} & & \cdots & & \mathbf{1} & 0 & \mathbf{1} \\
\hline
\end{array}\right\}}^{p+3} p+3
\qquad
\overbrace{\left.\begin{array}{|cccccc|c|}
\hline
\mathbf{1} & \mathbf{1} & 0 & & \cdots & & 0 \\
0 & \mathbf{1} & \mathbf{1} & 0 & \cdots & & 0 \\
 & & & \cdots & & & \\
0 & \cdots & & 0 & \mathbf{1} & \mathbf{1} & 0 \\
\hline
0 & \mathbf{1} & \cdots & & \mathbf{1} & 0 & \mathbf{1} \\
\hline
\end{array}\right\}}^{p+3} p+2
$$

M_{I_p}, $p \geq 1$ $\qquad$ M_{II_p}, $p \geq 1$ $\qquad$ M_{III_p}, $p \geq 1$

$$
\begin{array}{|cccccc|}
\hline
\mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\
\mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\
\hline
\end{array}
\qquad
\begin{array}{|ccccc|}
\hline
\mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\
0 & 0 & \mathbf{1} & \mathbf{1} & 0 \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\
\mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} \\
\hline
\end{array}
$$

M_{IV} $\qquad$ M_{V}

Theorem: A matrix has the C1P iff it contains none of the shown matrices.

[Tucker, Journal of Combinatorial Theory (B), 1972]

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

$$\overbrace{\left.\begin{array}{|cccccc|}\hline \mathbf{1} & \mathbf{1} & 0 & \cdots & & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & \cdots & 0 \\ & & & \ddots & & \\ 0 & \cdots & & 0 & \mathbf{1} & \mathbf{1} \\ \hline \mathbf{1} & 0 & \cdots & & 0 & \mathbf{1} \\ \hline \end{array}\right\}}^{p+2} p+2$$

M_{I_p}, $p \geq 1$

$$\overbrace{\left.\begin{array}{|cccccc|c|}\hline \mathbf{1} & \mathbf{1} & 0 & & \cdots & & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & \cdots & & 0 \\ & & & \ddots & & & \\ 0 & \cdots & & 0 & \mathbf{1} & \mathbf{1} & 0 \\ \hline 0 & \mathbf{1} & & \cdots & & & \mathbf{1} \\ \mathbf{1} & & \cdots & & \mathbf{1} & 0 & \mathbf{1} \\ \hline \end{array}\right\}}^{p+3} p+3$$

M_{II_p}, $p \geq 1$

$$\overbrace{\left.\begin{array}{|cccccc|c|}\hline \mathbf{1} & \mathbf{1} & 0 & & \cdots & & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & \cdots & & 0 \\ & & & \ddots & & & \\ 0 & \cdots & & 0 & \mathbf{1} & \mathbf{1} & 0 \\ \hline 0 & \mathbf{1} & \cdots & & \mathbf{1} & 0 & \mathbf{1} \\ \hline \end{array}\right\}}^{p+3} p+2$$

M_{III_p}, $p \geq 1$

$$\begin{array}{|cccccc|}\hline \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\ \hline \end{array}$$

M_{IV}

$$\begin{array}{|ccccc|}\hline \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} \\ \hline \end{array}$$

M_{V}

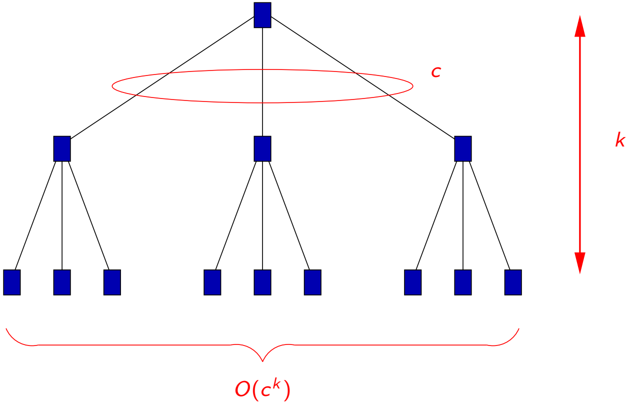
Approach: Use a search tree algorithm.

Repeat:

1. Search for a “forbidden submatrix”.
2. Branch on which of its columns has to be deleted.

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

Search Tree Algorithm:



Finite size c of forbidden matrices $\Rightarrow$ search tree of size $O(c^k)$.
(Alternatively: Factor-c approximation algorithm.)

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

$$\left.\overbrace{\begin{array}{|cccccc|}\hline \mathbf{1} & \mathbf{1} & 0 & \cdots & & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & \cdots & 0 \\ & & & \ddots & & \\ 0 & \cdots & & 0 & \mathbf{1} & \mathbf{1} \\ \hline \mathbf{1} & 0 & \cdots & & 0 & \mathbf{1} \\ \hline \end{array}}^{p+2}\right\} p+2$$

M_{I_p}, $p \geq 1$

$$\left.\overbrace{\begin{array}{|cccccc|c|}\hline \mathbf{1} & \mathbf{1} & 0 & & \cdots & & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & \cdots & & 0 \\ & & & \ddots & & & \\ 0 & \cdots & & 0 & \mathbf{1} & \mathbf{1} & 0 \\ \hline 0 & \mathbf{1} & & \cdots & & & \mathbf{1} \\ \mathbf{1} & & \cdots & & \mathbf{1} & 0 & \mathbf{1} \\ \hline \end{array}}^{p+3}\right\} p+3$$

M_{II_p}, $p \geq 1$

$$\left.\overbrace{\begin{array}{|cccccc|c|}\hline \mathbf{1} & \mathbf{1} & 0 & & \cdots & & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & \cdots & & 0 \\ & & & \ddots & & & \\ 0 & \cdots & & 0 & \mathbf{1} & \mathbf{1} & 0 \\ \hline 0 & \mathbf{1} & & \cdots & \mathbf{1} & 0 & \mathbf{1} \\ \hline \end{array}}^{p+3}\right\} p+2$$

M_{III_p}, $p \geq 1$

$$\begin{array}{|cccccc|}\hline \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\ \hline \end{array}$$

M_{IV}

$$\begin{array}{|ccccc|}\hline \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} \\ \hline \end{array}$$

M_{V}

A $(*, \Delta)$-matrix can contain

- M_{I_p} with unbounded size,
- M_{II_p} with $1 \leq p \leq \Delta - 2$,
- M_{III_p} with $1 \leq p \leq \Delta - 1$,
- M_{IV}, and M_{V}.

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

Problem: Matrices M_{I_p} of unbounded size can occur.

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

Problem: Matrices M_{I_p} of unbounded size can occur.

Idea: First destroy all “small” forbidden submatrices (search tree algorithm), and then see what happens...

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

Algorithmic framework for Min-COS-C / Min-COS-R:

1. Destroy the submatrices from

$$X := \{M_{\mathrm{I}_p} \mid 1 \leq p \leq \Delta - 1\} \cup \{M_{\mathrm{II}_p} \mid 1 \leq p \leq \Delta - 2\} \\ \cup \{M_{\mathrm{III}_p} \mid 1 \leq p \leq \Delta - 1\} \cup \{M_{\mathrm{IV}}, M_{\mathrm{V}}\}.$$

2. Destroy the remaining M_{I_p} $(p \geq \Delta)$.

We show:

- We can find a submatrix from X in polynomial time.
- If a $(*, \Delta)$-matrix M contains none of the matrices in X as a submatrix, then M can be divided into "independent" submatrices that have the *"circular ones property (Circ1P)"*.
- Min-COS-C / Min-COS-R can be solved in polynomial time on $(*, \Delta)$-matrices with the Circ1P.

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

Algorithmic framework for Min-COS-C / Min-COS-R:

1. Destroy the submatrices from

$$X := \{M_{\mathrm{I}_p} \mid 1 \leq p \leq \Delta - 1\} \cup \{M_{\mathrm{II}_p} \mid 1 \leq p \leq \Delta - 2\} \\ \cup \{M_{\mathrm{III}_p} \mid 1 \leq p \leq \Delta - 1\} \cup \{M_{\mathrm{IV}}, M_{\mathrm{V}}\}.$$

2. Destroy the remaining M_{I_p} $(p \geq \Delta)$.

We show:

- **We can find a submatrix from X in polynomial time.**
- If a $(*, \Delta)$-matrix M contains none of the matrices in X as a submatrix, then M can be divided into "independent" submatrices that have the *"circular ones property (Circ1P)"*.
- Min-COS-C / Min-COS-R can be solved in polynomial time on $(*, \Delta)$-matrices with the Circ1P.

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

Algorithmic framework for Min-COS-C / Min-COS-R:

1. Destroy the submatrices from

$$X := \{M_{\mathrm{I}_p} \mid 1 \leq p \leq \Delta - 1\} \cup \{M_{\mathrm{II}_p} \mid 1 \leq p \leq \Delta - 2\} \cup \{M_{\mathrm{III}_p} \mid 1 \leq p \leq \Delta - 1\} \cup \{M_{\mathrm{IV}}, M_{\mathrm{V}}\}.$$

2. Destroy the remaining M_{I_p} $(p \geq \Delta)$.

We show:

- We can find a submatrix from X in polynomial time.
- **If a $(*, \Delta)$-matrix M contains none of the matrices in X as a submatrix, then M can be divided into "independent" submatrices that have the "circular ones property (Circ1P)".**
- Min-COS-C / Min-COS-R can be solved in polynomial time on $(*, \Delta)$-matrices with the Circ1P.

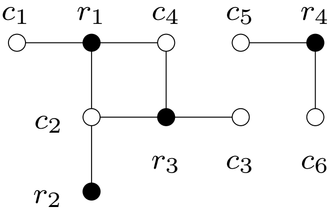
Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

If a $(*, \Delta)$-matrix M contains none of the matrices in X as a submatrix, then every component of M has the *circular ones property (Circ1P)*.

[Dom, Guo, Niedermeier, TAMC '07]

Components of a matrix:

$$\begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} \begin{array}{cccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ \hline \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \end{array}$$



Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

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[Dom, Guo, Niedermeier, TAMC '07]

$$\begin{array}{ccccccccccc}
1 & 1 & 1 & 1 & 1 & & & & & & \\
 & 1 & 1 & 1 & 1 & 1 & & & & & \\
 & & & & 1 & 1 & 1 & 1 & & & \\
 & & & & & & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & & & & & & & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & & & & & & & 1
\end{array}$$

A 0/1-matrix M has the Circ1P if its columns can be permuted such that in each row the 1's form a block when M is wrapped around a vertical cylinder.

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

If a $(*, \Delta)$-matrix M contains none of the matrices in X as a submatrix, then every component of M has the *circular ones property (Circ1P)*.

[Dom, Guo, Niedermeier, TAMC '07]

Proof by contraposition:

If a component B of a $(*, \Delta)$-matrix does not have the Circ1P, then it contains one of the submatrices from X.

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

Theorem: Let M be a matrix and c_j be a column of M. Form the matrix M' from M by complementing all rows with a 1 in column c_j. Then M has the Circ1P iff M' has the C1P.

[Tucker, Pacific Journal of Mathematics, 1971]

$$
\begin{array}{ccccccccccc}
1 & 1 & 1 & 1 & 1 & & & & & & \\
 & 1 & 1 & 1 & 1 & 1 & & & & & \\
 & & & & 1 & 1 & 1 & 1 & & & \\
 & & & & & & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & & & & & & & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & & & & & & & 1
\end{array}
$$

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

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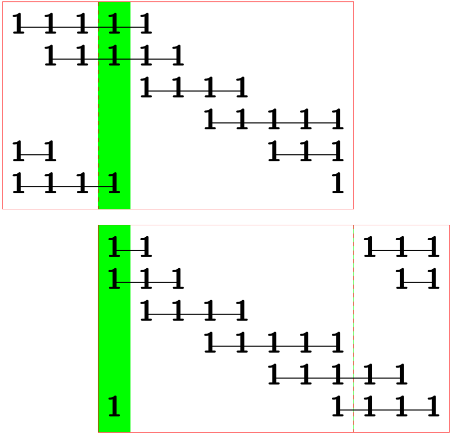

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

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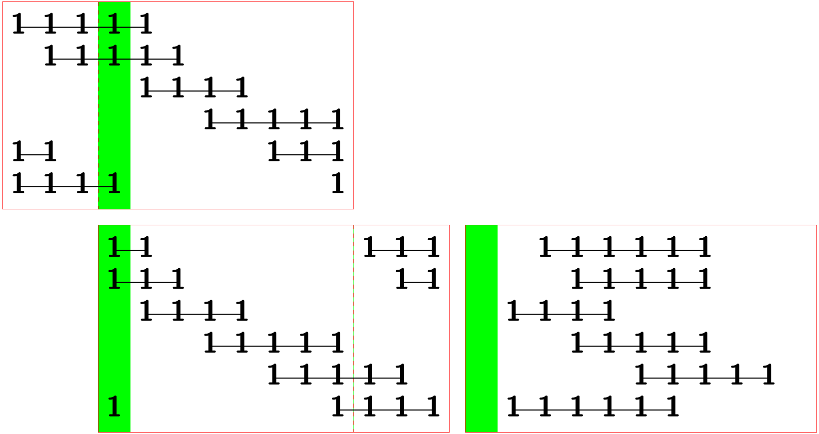
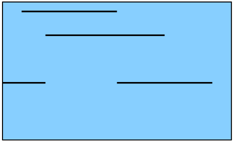


Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

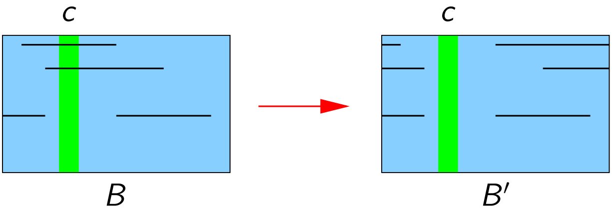
Component B without circular ones property.



B

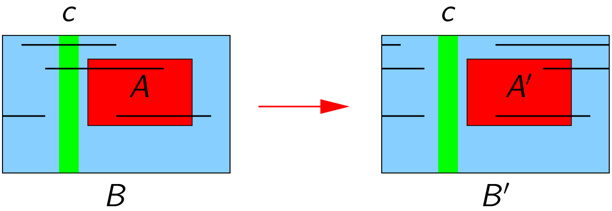
Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

Component B without circular ones property.
$\Rightarrow \exists$ column c such that B' does not have the C1P.



Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

Component B without circular ones property.
$\Rightarrow \exists$ column c such that B' does not have the C1P.
$\Rightarrow$ There is a forbidden submatrix A' in B'.



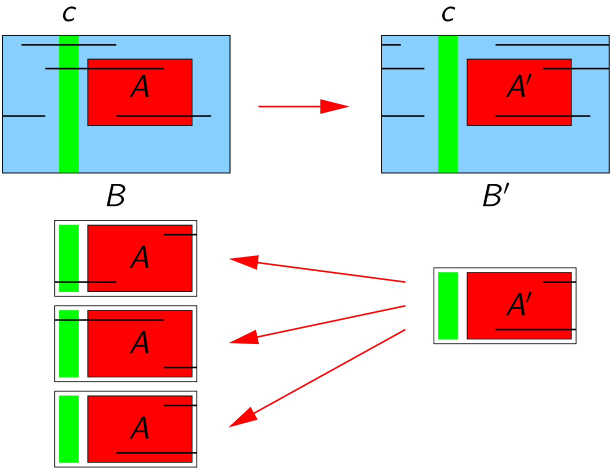
Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

Component B without circular ones property.

$\Rightarrow \exists$ column c such that B' does not have the C1P.

$\Rightarrow$ There is a forbidden submatrix A' in B'.

$\Rightarrow$ We can always find a submatrix from X in B.



Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

$$
\overbrace{\left.\begin{array}{|ccccc|}\hline \mathbf{1} & \mathbf{1} & 0 & \cdots & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0\cdots & 0 \\ & & \cdots & & \\ 0 & \cdots & 0 & \mathbf{1} & \mathbf{1} \\ \hline \mathbf{1} & 0 & \cdots & 0 & \mathbf{1} \\ \hline \end{array}\right\}p+2}^{p+2}
\qquad
\overbrace{\left.\begin{array}{|ccccc|c|}\hline \mathbf{1} & \mathbf{1} & 0 & & \cdots & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & \cdots & 0 \\ & & \cdots & & & \\ 0 & \cdots & 0 & \mathbf{1} & \mathbf{1} & 0 \\ \hline 0 & \mathbf{1} & & \cdots & & \mathbf{1} \\ \mathbf{1} & & \cdots & \mathbf{1} & 0 & \mathbf{1} \\ \hline \end{array}\right\}p+3}^{p+3}
\qquad
\overbrace{\left.\begin{array}{|ccccc|c|}\hline \mathbf{1} & \mathbf{1} & 0 & & \cdots & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & \cdots & 0 \\ & & \cdots & & & \\ 0 & \cdots & 0 & \mathbf{1} & \mathbf{1} & 0 \\ \hline 0 & \mathbf{1} & \cdots & \mathbf{1} & 0 & \mathbf{1} \\ \hline \end{array}\right\}p+2}^{p+3}
$$

$M_{\mathrm{I}_p},\ p \geq 1$ $\qquad$ $M_{\mathrm{II}_p},\ p \geq 1$ $\qquad$ $M_{\mathrm{III}_p},\ p \geq 1$

$$
\begin{array}{|cccccc|}\hline \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\ \hline \end{array}
\qquad
\begin{array}{|ccccc|}\hline \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} \\ \hline \end{array}
$$

M_{IV} $\qquad$ M_{V}

Case study 1: A' is an M_{IV}, row 2 has been complemented.

$$
\begin{array}{c|c|cccccc|}
 & & 1 & 2 & 5 & & 3 & 4 \\ \hline
1 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\
3 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{1} \\
2 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\
4 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\ \hline
\end{array}
\longleftarrow
\underbrace{\begin{array}{|c|cccccc|}\hline
0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\
0 & \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\ \hline
\end{array}}_{A'}
$$

Then we can find an M_{V} in B.

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

$$\left.\overbrace{\begin{array}{|cccccc|}\hline 1 & 1 & 0 & \cdots & & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ & & \ddots & & & \\ 0 & \cdots & & 0 & 1 & 1 \\ \hline 1 & 0 & \cdots & & 0 & 1 \\ \hline \end{array}}^{p+2}\right\} p+2$$

M_{I_p}, $p \geq 1$

$$\left.\overbrace{\begin{array}{|cccccc|c|}\hline 1 & 1 & 0 & & \cdots & & 0 \\ 0 & 1 & 1 & 0 & \cdots & & 0 \\ & & & \ddots & & & \\ 0 & \cdots & & 0 & 1 & 1 & 0 \\ \hline 0 & 1 & & & \cdots & & 1 \\ 1 & & \cdots & & 1 & 0 & 1 \\ \hline \end{array}}^{p+3}\right\} p+3$$

M_{II_p}, $p \geq 1$

$$\left.\overbrace{\begin{array}{|cccccc|c|}\hline 1 & 1 & 0 & & \cdots & & 0 \\ 0 & 1 & 1 & 0 & \cdots & & 0 \\ & & & \ddots & & & \\ 0 & \cdots & & 0 & 1 & 1 & 0 \\ \hline 0 & 1 & & \cdots & 1 & 0 & 1 \\ \hline \end{array}}^{p+3}\right\} p+2$$

M_{III_p}, $p \geq 1$

$$\begin{array}{|cccccc|}\hline 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ \hline \end{array}$$

M_{IV}

$$\begin{array}{|ccccc|}\hline 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ \hline \end{array}$$

M_{V}

Case study 2: A' is an M_{V}, row 3 has been complemented.

$$\begin{array}{c|c|ccccc|} & 6 & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 1 \\ 4 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline \end{array} \longleftarrow \begin{array}{|c|ccccc|}\hline 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 \\ \hline \end{array}$$

A'

Then we can find an M_{IV} in B.

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

Most complicated case: A' is an M_{I_p} with $p \geq \Delta$.

$$A$$

$$\begin{array}{ccccccc}
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
\hline
\mathbf{1} & & 0 & \mathbf{1} & \mathbf{1} & 0 &
\end{array}$$

$$A'$$

$$\begin{array}{ccccccc}
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1
\end{array}$$

$$A$$

$$\begin{array}{c|ccccccc}
 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\mathbf{1} & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
\hline
\mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Then we can find an M_{III_1} or an M_{IV} in B.

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

Algorithmic framework for Min-COS-C / Min-COS-R:

1. Destroy the submatrices from

$$X := \{M_{\mathrm{I}_p} \mid 1 \le p \le \Delta - 1\} \cup \{M_{\mathrm{II}_p} \mid 1 \le p \le \Delta - 2\} \\ \cup \{M_{\mathrm{III}_p} \mid 1 \le p \le \Delta - 1\} \cup \{M_{\mathrm{IV}}, M_{\mathrm{V}}\}.$$

2. Destroy the remaining M_{I_p} $(p \ge \Delta)$.

We show:

- We can find a submatrix from X in polynomial time.
- If a $(*, \Delta)$-matrix M contains none of the matrices in X as a submatrix, then M can be divided into "independent" submatrices that have the *"circular ones property (Circ1P)"*.
- **Min-COS-C / Min-COS-R can be solved in polynomial time on $(*, \Delta)$-matrices with the Circ1P.**

From Circ1P to C1P

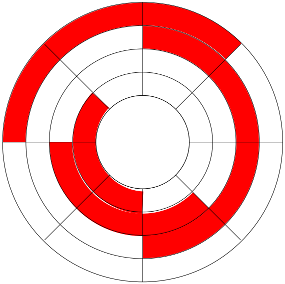
C1P:	1's blockwise after column permutations
Circ1P:	1's blockwise on a cylinder after column permutations
strong C1P:	1's blockwise *without* column permutations
strong Circ1P:	1's blockwise on a cylinder *without* column permutations

(Circ1P/C1P means: Strong Circ1P/strong C1P can be obtained by column permutations.)

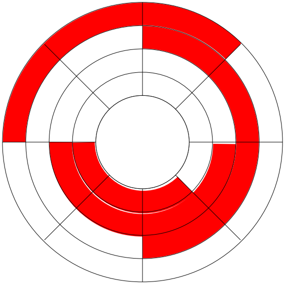
From Circ1P to C1P

We imagine the matrices as wrapped around a vertical cylinder.

Strong Circ1P:



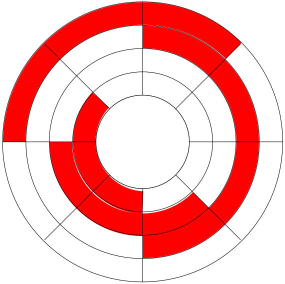
Strong C1P:



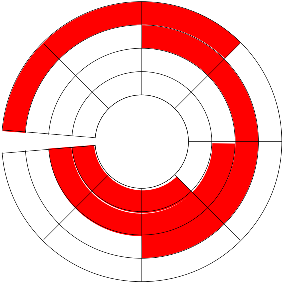
From Circ1P to C1P

We imagine the matrices as wrapped around a vertical cylinder.

Strong Circ1P:



Strong C1P:



Strong C1P =
Strong Circ1P + “cut”

From Circ1P to C1P

Our task:

strong Circ1P	$\xrightarrow{}$ column deletions	strong Circ1P + C1P

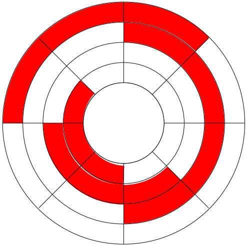
From Circ1P to C1P

Our task:

strong Circ1P	$\xrightarrow{\text{column deletions}}$	strong Circ1P + C1P

First consider this task:

strong Circ1P	$\xrightarrow{\text{column deletions}}$	strong Circ1P + strong C1P



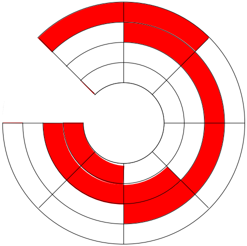
From Circ1P to C1P

Our task:

strong Circ1P	$\xrightarrow{\text{column deletions}}$	strong Circ1P + C1P

First consider this task:

strong Circ1P	$\xrightarrow{\text{column deletions}}$	strong Circ1P + strong C1P



Obs.: Deleting a consecutive set of columns is always optimal.

From Circ1P to C1P

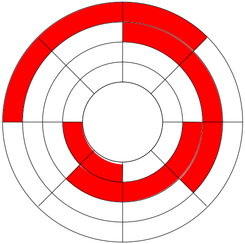
Our task:

strong Circ1P	$\xrightarrow[\text{column deletions}]{}$	strong Circ1P + C1P

First consider this task: Easy!!!

strong Circ1P	$\xrightarrow[\text{column deletions}]{}$	strong Circ1P + strong C1P

We hope: Does "strong Circ1P + C1P" imply "strong C1P"?



From Circ1P to C1P

Conjecture: If a matrix has the strong Circ1P and the C1P, then it has also the strong C1P.

From Circ1P to C1P

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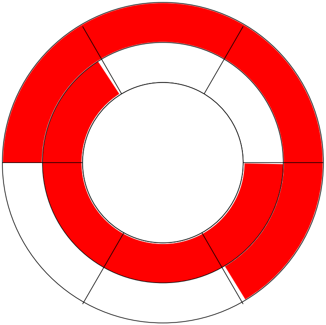
Counterexample:



From Circ1P to C1P

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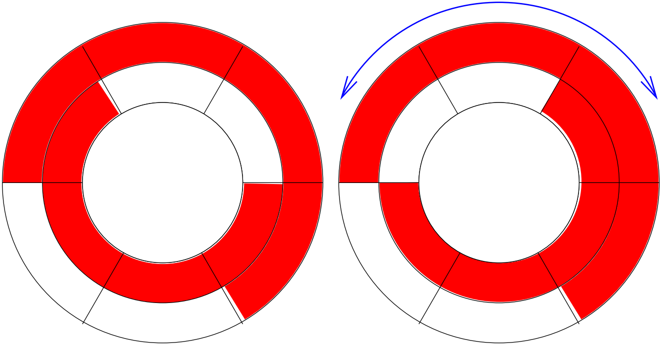
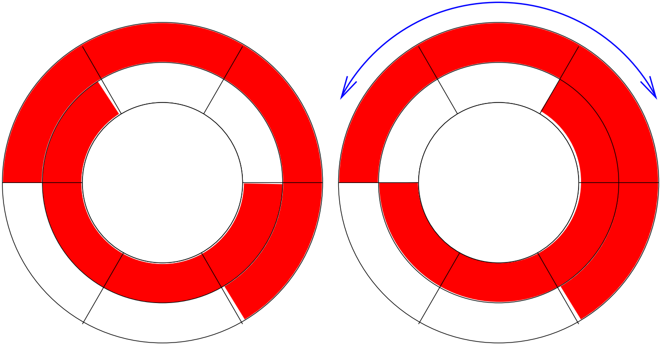


From Circ1P to C1P

Conjecture: If a matrix has the strong Circ1P and the C1P, then it has also the strong C1P.

Counterexample:



New conjecture: If a matrix with $\geq 2\Delta - 1$ columns has the strong Circ1P and the C1P, then it has also the strong C1P.

From Circ1P to C1P

To be proven: If a matrix with $\geq 2\Delta - 1$ columns has the strong Circ1P and the C1P, then it has also the strong C1P.

Very helpful:
Theorem: Let M have the strong Circ1P. Then *every* column permutation that also yields the strong Circ1P can be obtained by a series of circular module reversals.
[Hsu, McConnell, Theor. Comput. Sci., 2003]

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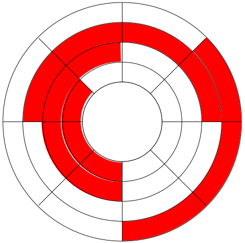

strong Circ1P+ C1P

From Circ1P to C1P

To be proven: If a matrix with $\geq 2\Delta - 1$ columns has the strong Circ1P and the C1P, then it has also the strong C1P.

Very helpful:
Theorem: Let M have the strong Circ1P. Then *every* column permutation that also yields the strong Circ1P can be obtained by a series of circular module reversals.
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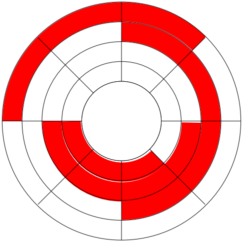
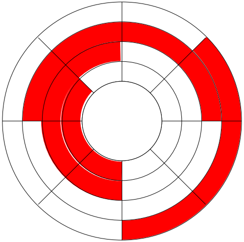


strong Circ1P+ strong C1P



strong Circ1P+ C1P

From Circ1P to C1P

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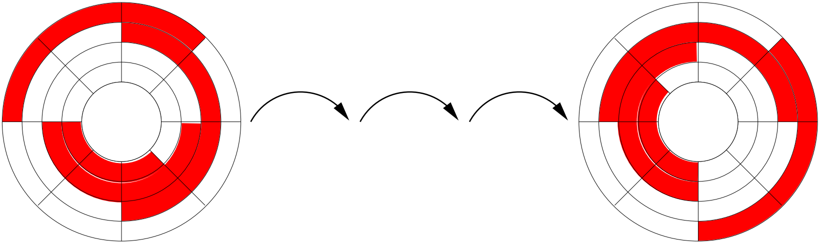


strong Circ1P+ strong C1P

strong Circ1P+ C1P

From Circ1P to C1P

Now to be proven: Let M be a matrix with with $\geq 2\Delta - 1$ columns that has the strong Circ1P and the strong C1P. Reversing an arbitrary circular module of M does not affect these properties.

From Circ1P to C1P

Algorithm for Min-COS-C on matrices with Circ1P:

1. Permute the columns to get the strong Circ1P.
2. Search for a set of *consecutive* consecutive columns whose deletion yields the strong C1P.

[Dom, Niedermeier, ACiD '07]

Results for Min-COS-C and Min-COS-R

FPT algorithm:

Running time:

$(\lvert\text{submatrix}\rvert)^k$	$\cdot$	(search	$+$	"Circ1P$\rightarrow$C1P" time)
$(\Delta+2)^k$	$\cdot$	$(n^{O(1)}$	$+$	$O(\Delta mn))$

Approximation algorithm:

Approximation factor: $\lvert\text{submatrix}\rvert$

Running time: $k \cdot$ (search $+$ "Circ1P$\rightarrow$ C1P" time)

Open Question

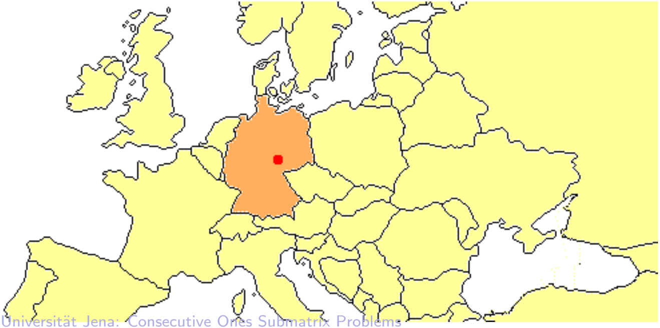
How can a matrix that has the (strong) Circ1P be modified by deleting a minimum number of 1-entries such that the resulting matrix has the C1P?

More Open Questions

(1's per col, 1's per row)	Max-COS-C	Min-COS-C
$(3, 2)$	0.5-approx[1]	
$(*, 2)$	• No const. approx.[1] • W[1]-hard	• No 2,72-approx. • Problem kernel
$(*, \Delta)$	• No const. approx.[1] • W[1]-hard	• $(\Delta + 2)$-approx. • $O((\Delta+2)^k \cdot \Delta^{O(\Delta)} \cdot \lvert M\rvert^{O(1)})$-alg.
$(2, 3)$	0.8-approx[1]	
$(2, *)$	0.5-approx[1]	• 6-approx • $O(6^k \cdot \mathrm{pol}(\lvert M\rvert))$-alg.
$(\Delta, *)$	?	?

[1] [Tan, Zhang, Algorithmica, 2007]

Jena, Germany

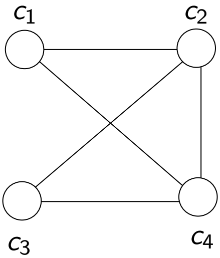


Min-COS-C on $(*, 2)$-Matrices

Min-COS-C is equivalent to Induced Disjoint Paths Subgraph (IDPS).

Induced Disjoint Paths Subgraph (IDPS)
Given: A graph G and a positive integer k.
Question: Can we delete at most k vertices of G such that the resulting graph is a vertex-disjoint disjoint union of paths?

c_1	c_2	c_3	c_4
1	0	0	1
1	1	0	0
0	0	1	1
0	1	1	0
0	1	0	1



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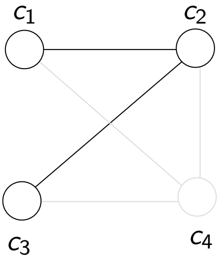
$$\begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \\ \hline 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}$$



Problem Kernel for Min-COS-C on $(*, 2)$-Matrices

Problem Kernel:
Given a parameterized problem instance (X, k).
Transform it in polynomial time into an instance (X', k') with $|X'| \leq f(k)$ and $k' \leq k$.

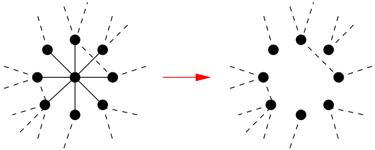
Problem Kernel for Min-COS-C on $(*, 2)$-Matrices

Theorem: IDPS with parameter k admits a problem kernel with $O(k^2)$ vertices and $O(k^2)$ edges.

Data reduction rules:

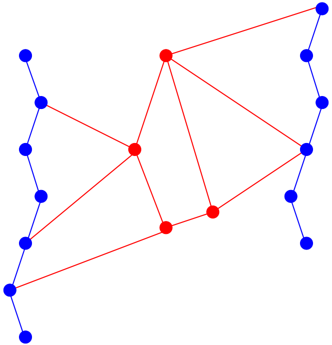
1. If a degree-two vertex v has two degree-at-most-two neighbors u, w with $\{u, w\} \notin E$, remove v from G and connect u, w by an edge.

2. If a vertex v has more than $k + 2$ neighbors, then remove v from G, add v to the solution, and decrease k by one.



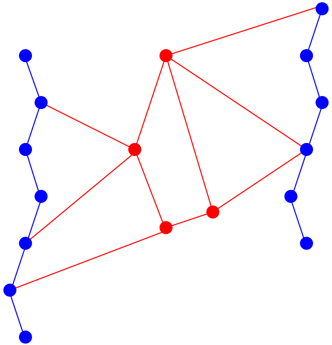
Problem Kernel for Min-COS-C on $(*, 2)$-Matrices

- At most k red vertices.



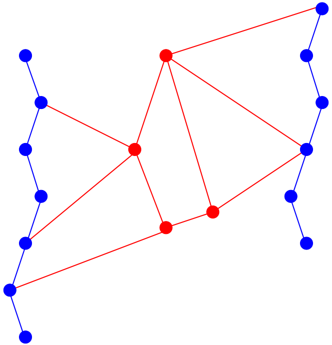
Problem Kernel for Min-COS-C on $(*, 2)$-Matrices

- At most k red vertices.
- They have at most $k \cdot (k+2)$ blue neighbors.



Problem Kernel for Min-COS-C on $(*, 2)$-Matrices

- At most k red vertices.
- They have at most $k \cdot (k + 2)$ blue neighbors.
- At least every third blue vertex must be a neighbor of a red vertex.



Problem Kernel for Min-COS-C on $(*, 2)$-Matrices

- At most k red vertices.
- They have at most $k \cdot (k+2)$ blue neighbors.
- At least every third blue vertex must be a neighbor of a red vertex.

$\Rightarrow k + 3 \cdot k \cdot (k+2)$ vertices.

$\Rightarrow k \cdot (k+2) + 3 \cdot k \cdot (k+2) - 1$ edges.