Bounded Degree Closest k-Tree Power is NP-Complete

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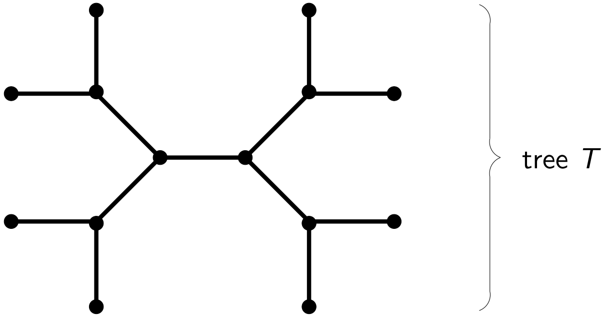
Structure of the Talk

- **Introduction: Tree power and tree root problems**
- A reduction from 3-Vertex Cover to Closest k-Tree Power with Bounded Degree

k-Tree Roots and k-Tree Powers

Example: A tree network consisting of 14 processors. Passing information from one processor to the next requires one timestep.



Black edges: Communication possible in one timestep

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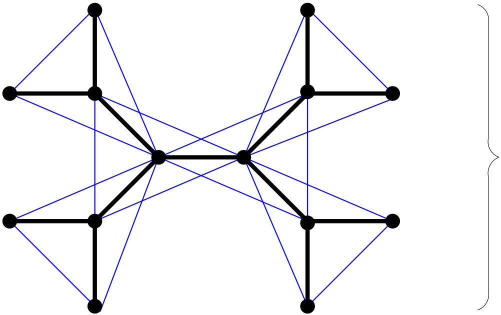

2-tree power T^2

Black edges: Communication possible in one timestep
Blue edges: Communication possible in two timesteps

k-Tree Roots and k-Tree Powers

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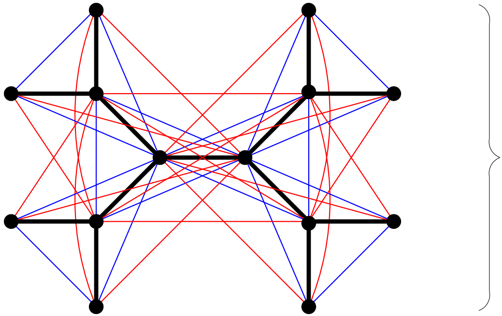
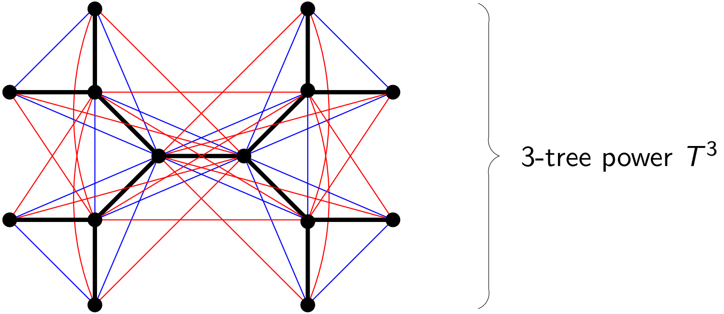


Black edges: Communication possible in one timestep
Blue edges: Communication possible in two timesteps
Red edges: Communication possible in three timesteps

k-Tree Roots and k-Tree Powers



Definition

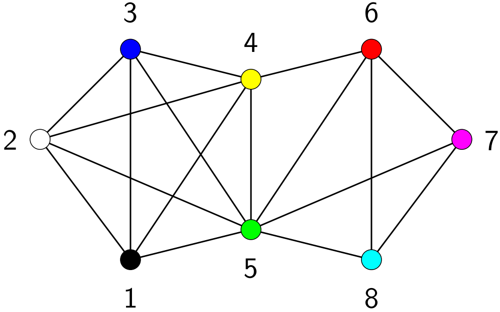
A graph $G = (V, E_G)$ is a *k-tree power* if there is a tree $T = (V, E_T)$ with

$$\forall u, v \in V : \; \mathrm{dist}_T \leq k \;\Leftrightarrow\; (u, v) \in E_G.$$

T is called the *k-tree root* of G.

Tree power recognition / Computing tree roots

Example: Constructing phylogenies.
Given: A graph with node set V—denoting biological species—and an edge set denoting similarities between the species.
Question: Is there a tree with node set V in which two nodes u, v have distance at most k iff they are connected in the given graph?



Set of species—edges denote similarities

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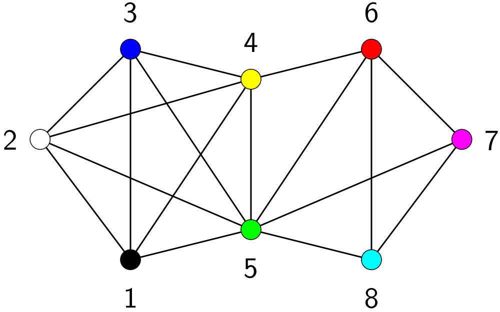

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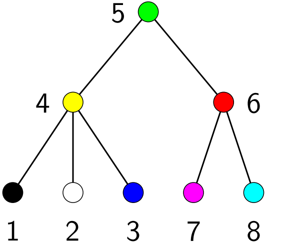


Phylogeny: 2-tree root

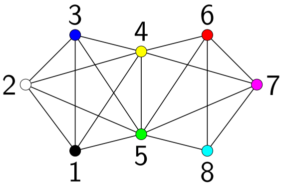
Tree power recognition / Computing tree roots

k-Tree Power
Instance: A graph G.
Question: Is there a tree T with $T^k = G$?

- Solvable in linear time for $k = 2$
  [Y. L. Lin, S. S. Skiena, *SIAM Journal on Discrete Mathematics*, 1995]
- Solvable in polynomial time for every $k \geq 3$
  [P. E. Kearney, D. G. Corneil, *Journal of Algorithms*, 1998]
- NP-complete for every $k \geq 2$ if T may be an arbitrary graph (Graph Power problem)
  [R. Motwani, M. Sudan, *Discrete Applied Mathematics*, 1994]

A graph modification problem

What to do if a given graph has no k-tree root?



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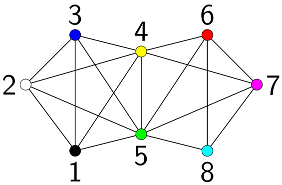

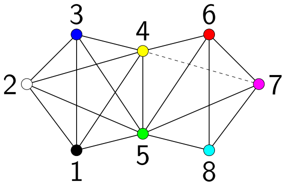

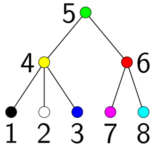

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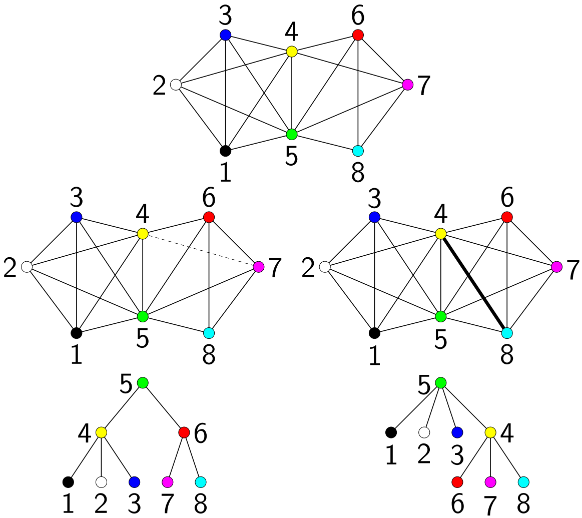

A graph modification problem

CLOSEST k-TREE POWER (CTPk)
Instance: A graph G.
Question: Is there a k-tree power T^k such that T^k and G differ by at most ℓ edges: $|(E(T^k) \setminus E(G)) \cup (E(G) \setminus E(T^k))| \leq \ell$?

- NP-complete for every $k \geq 2$
  [P. E. Kearney, D. G. Corneil, *Journal of Algorithms*, 1998]

CLOSEST k-TREE POWER WITH BOUNDED DEGREE (Δ-CTPk)
The maximum vertex degree of T is at most Δ.

- Complexity open so far for every $k, \Delta \geq 2$. Open question at COCOON 2004! [T. Tsukiji, Z.-Z. Chen, *Proc. 10th COCOON*, 2004]

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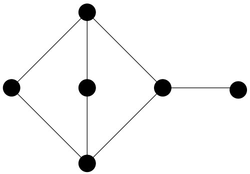
- Introduction: Tree power and tree root problems
- **A reduction from 3-Vertex Cover to Closest k-Tree Power with Bounded Degree**

Idea of the reduction

We reduce the NP-complete [M. R. Garey, D. S. Johnson, L. J. Stockmeyer, *Theoretical Computer Science*, 1976] problem 3-VERTEX COVER to Δ-CTPk with $\Delta \geq 4$.

3-VERTEX COVER
Input: A graph $G = (V, E)$ with a maximum vertex degree 3 and a nonnegative integer ℓ.
Question: Is there a set $C \subseteq V$ of at most ℓ vertices such that each edge from E has at least one endpoint in C?



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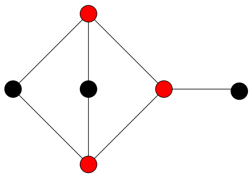
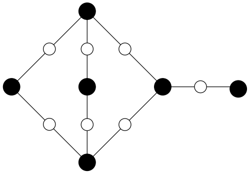


“Edge modification version” of 3-Vertex Cover

Reformulation of 3-Vertex Cover:

- We insert an additional node in the middle of each edge.
- The question now is: Delete edges such that
  1. each “additional” node is adjacent to exactly one “original” vertex, and
  2. the number of “original” vertices having degree at least one is minimized.



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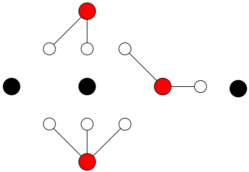

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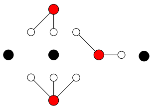

- We need $|E|$ edge deletions to transform each “additional” node into a degree-1-node—independent of the number of red vertices.

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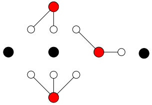

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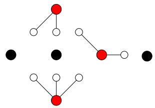

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- $\Rightarrow$ Replace the “original” vertices of the 3-Vertex Cover instance by gadgets such that for each red vertex further edge modifications are necessary.

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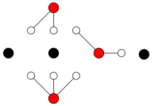

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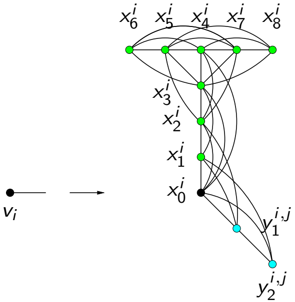
- $\Rightarrow$ Replace the “original” vertices of the 3-VERTEX COVER instance by gadgets such that for each red vertex further edge modifications are necessary.
- Replace the edges of the 3-VERTEX COVER instance (and the “additional” nodes) by gadgets.

Construction of a vertex gadget (case $k = 3$)

For each "original" vertex $v_i \in V$, $i = 1, \ldots, n$ of the 3-Vertex Cover instance $G = (V, E)$ we insert into the Δ-CTPk instance G_{CTP} a *vertex gadget*:



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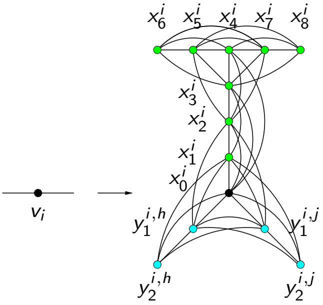

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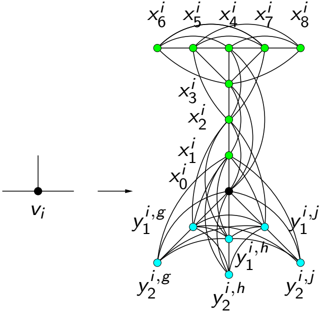
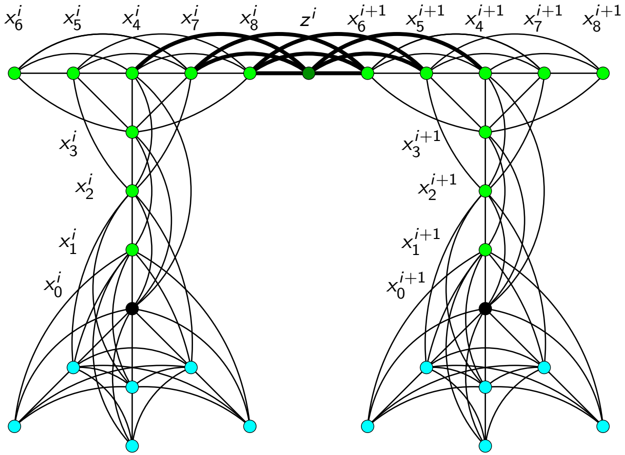


Making G_{CTP} a connected component (case $k = 3$)

To guarantee that G_{CTP} is connected we add $n - 1$ *connecting nodes* $z^i, \ldots, z^{n-1}$, and for all $1 \leq i < n$ we connect the gadgets of v_1 and v_{i+1}:



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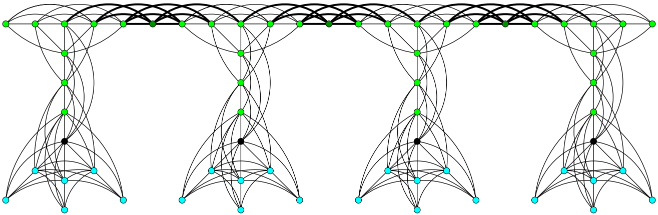
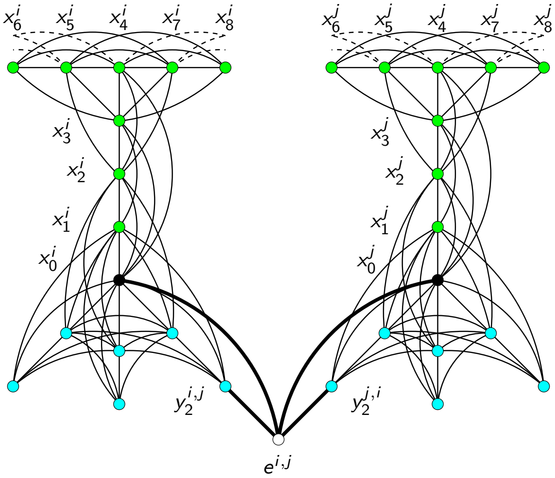
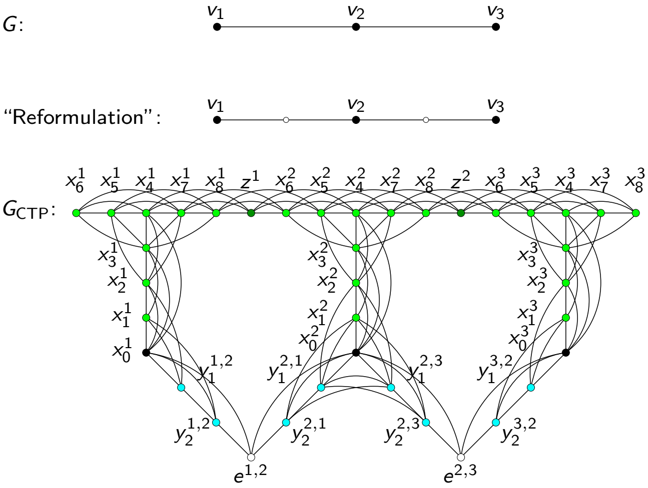


Construction of an edge gadget (case $k = 3$)

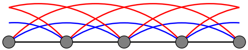
For each edge (v_i, v_j) of the 3-Vertex Cover instance $G = (V, E)$ we insert into the Δ-CTPk instance G_{CTP} an *edge gadget* consisting of an *edge node* $e^{i,j}$ and four edges:



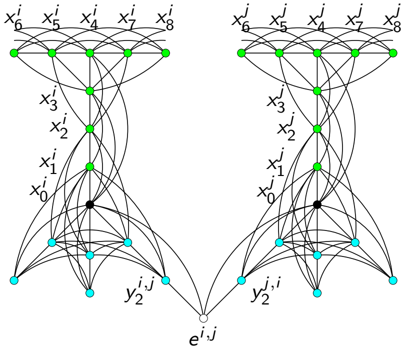
Example (case $k = 3$)



Observation for edge gadgets (case $k = 3$)

Structure of a 3-tree power: 

In order to obtain a 3-tree power, we have to insert one edge and to delete two edges in G_{CTP} for every edge in G:



Altogether: $3 \cdot |E|$ edge modifications.

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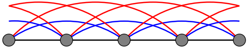
Structure of a 3-tree power: 

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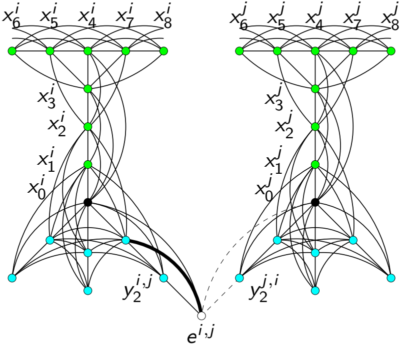

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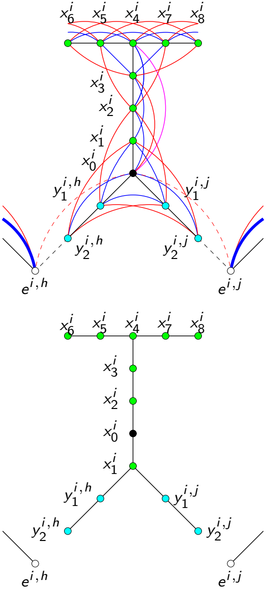
A detailed look on vertex gadgets (case $k = 3$)



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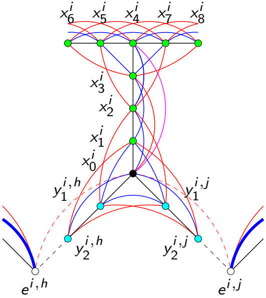

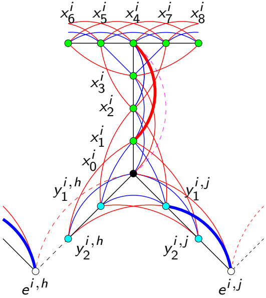

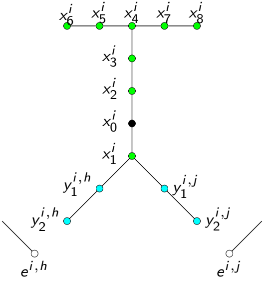

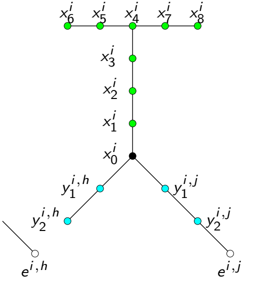

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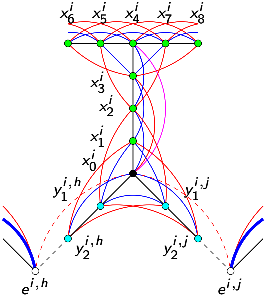
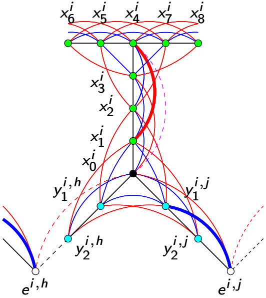


Vertex gadget connected to no edge node:

No edge modification in the vertex gadget

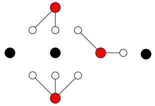


Vertex gadget connected to at least one edge node:

Two edge modifications in the vertex gadget

Counting the edge modifications

Altogether:
#modified edges $=$
$3 \cdot |E| + 2 \cdot$ #vertex gadgets connected to edge nodes



$\Rightarrow$ Number of vertex gadgets corresponding to red vertices are minimized.

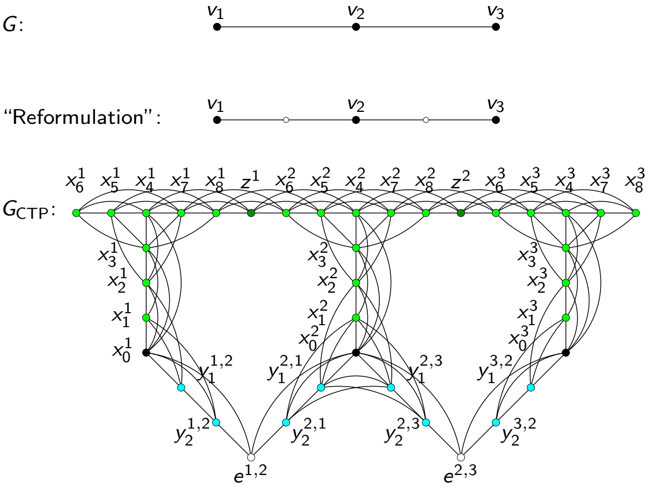
Theorem

G has a vertex cover of size x

$\Leftrightarrow$

G_{CTP} has a solution of size $3 \cdot |E| + 2 \cdot x$

Example (case $k = 3$)



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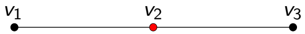
G:



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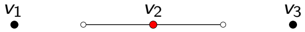
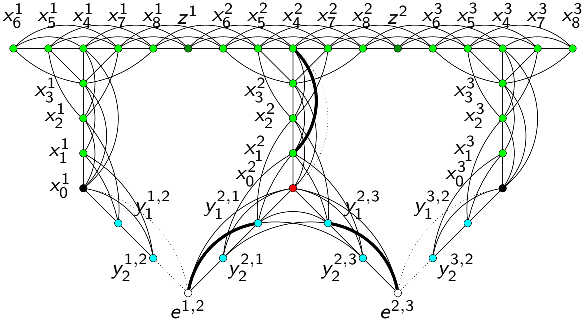


Solution:



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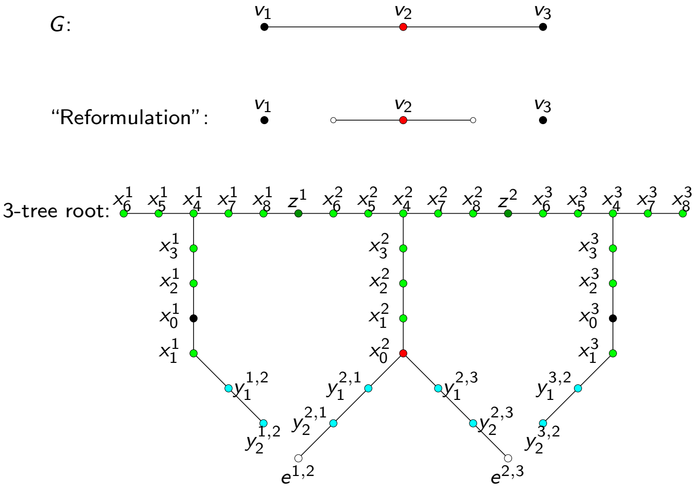

Open questions

- NP-completeness is only shown for $\Delta \geq 4$. What about $\Delta = 3$?
- What about the hardness if only edge deletions/insertions are allowed?
- Approximation or fixed-parameter tractability results for (Δ)-$\mathrm{CTP}k$?