Fixed-Parameter Tractability Results for Feedback Set Problems in Tournaments

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Structure of the Talk

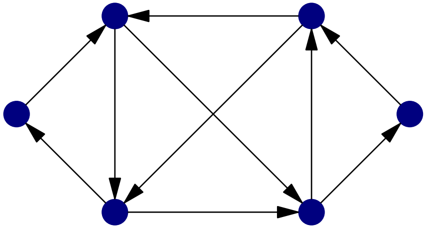
- The Feedback Vertex Set Problem in Tournaments
- An Iterative Compression Algorithm
- Further Results

The Problem Feedback Vertex Set

Definition (Feedback Vertex Set (FVS))

Input: Directed graph $G = (V, E)$, integer $k \geq 0$.

Output: Is there a subset $X \subseteq V$ of at most k vertices such that $G[V \setminus X]$ has no cycles?



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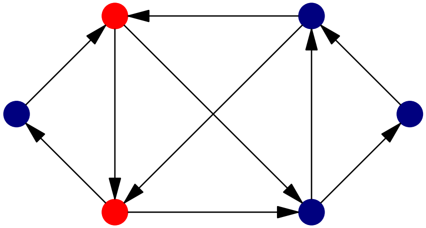

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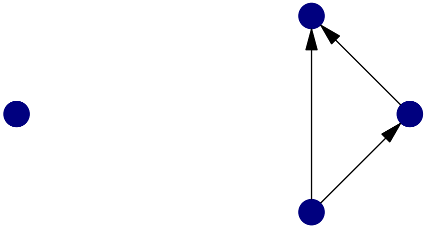
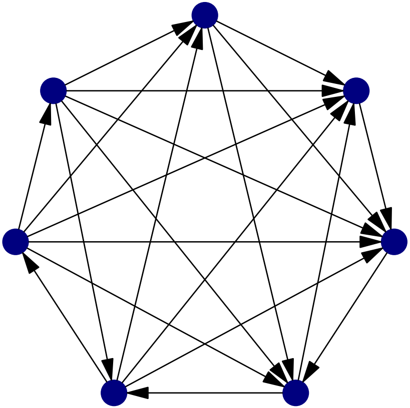


Complexity of FEEDBACK VERTEX SET

FEEDBACK VERTEX SET is NP-complete on:

- Directed graphs
  [Karp, 1972]
- Planar directed graphs
  [Yannakakis, STOC 1978]
- Directed graphs with in-/outdegree ≤ 2
  [Garey and Johnson, Computers and Intractability, 1979]
- Planar directed graphs with in-/outdegree ≤ 3
  [Garey and Johnson, Computers and Intractability, 1979]
- Tournaments
  [Speckenmeyer, WG 1989]

Tournaments



For each pair u, v of vertices, there is exactly one of the edges (u, v) and (v, u).

Approximability of Feedback Vertex Set

- Feedback Vertex Set is APX-hard on directed graphs.
  [Kann, PhD thesis, 1992]

- Known approximation ratios:

Undirected graphs	2	[Bafna et al., SIAM J. Disc. Math., 1999]
Digraphs	$O(\log n \log\log n)$	[Even et al., Algorithmica, 1998]
Planar digraphs	2.25	[Goemans and Williamson, IPCO 1996]
Tournaments	2.5	[Cai et al., SIAM J. Comput., 2001]

- Lower bound for FVS in tournaments: 1.36 unless P = NP.
  [Speckenmeyer, WG 1989, Dinur and Safra, Ann. of Math., 2005]

Fixed-Parameter Tractability (1)

A problem (G, k) is fixed-parameter tractable (FPT) $\Leftrightarrow$
(G, k) is solvable in time $f(k) \cdot n^{O(1)}$.

Is FEEDBACK VERTEX SET with parameter "solution size" FPT?

- Undirected graphs: Yes.
  [Bodlaender, Internat. J. Found. Comput. Sci., 1994;
  Downey and Fellows, Congressus Numerantium, 1992]
- Directed graphs: Open problem.
- Tournaments: Yes.
  [Raman and Saurabh, Theoret. Comput. Sci., 2006]

Fixed-Parameter Tractability (2)

Lemma

A tournament contains a cycle iff it contains a triangle.

$\Rightarrow$ Reformulation of the task: Destroy all triangles.

$\Rightarrow$ FEEDBACK VERTEX SET in tournaments can be reduced to 3-HITTING SET, which can be solved in time $O(2.18^k \cdot n^{O(1)})$ [Fernau, ECCC, 2004].

Our algorithm: Running time $O(2^k \cdot n^{O(1)})$.

Iterative Compression

"Compression routine":
Given a solution of size $k+1$, compute a solution of size k.
[Reed, Smith, and Vetta, Oper. Res. Lett., 2004]

Iterative Compression framework for FEEDBACK VERTEX SET:

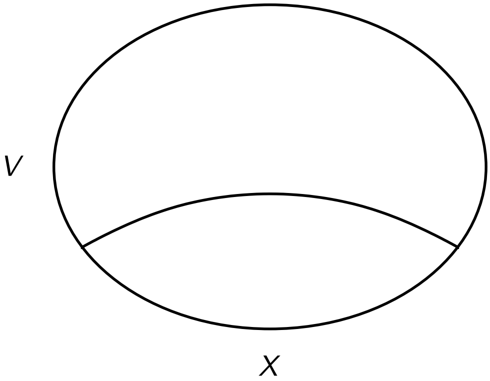
Start with an empty graph G and an empty solution X.
Repeat n times:
    Add a vertex v to G and to X.
    Try to compress X.
    If $|X| > k$, answer ''no''.

Invariant during the loop:
X is a solution of size at most k for the current graph G.

Compression Routine (1)

Task:
Given a solution X of size $k+1$, compute a solution X' of size k.

Approach:
Try all 2^{k+1} partitions of X into two subsets S and $X \setminus S$.



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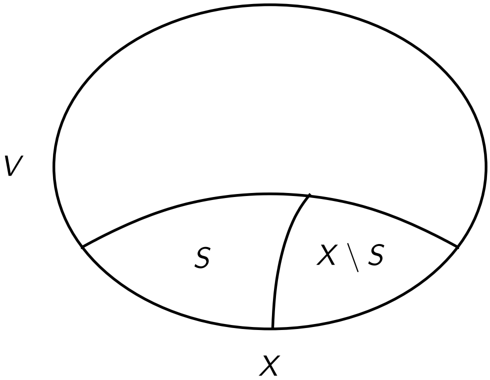

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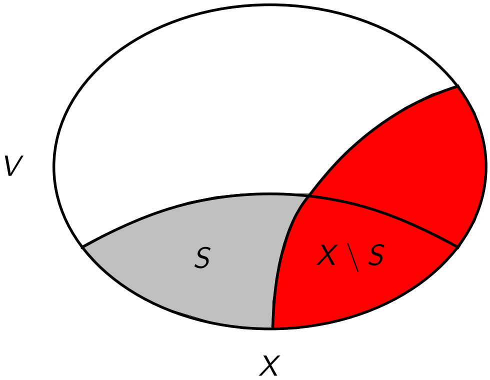

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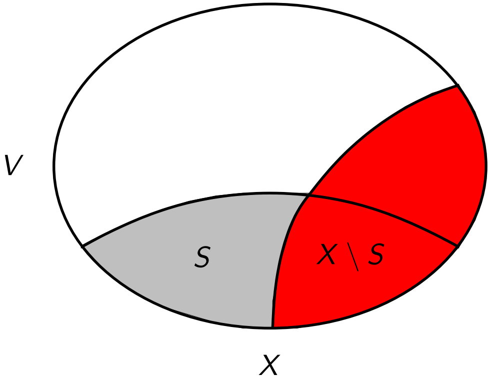

Vertices from $X \setminus S$ can be deleted immediately.

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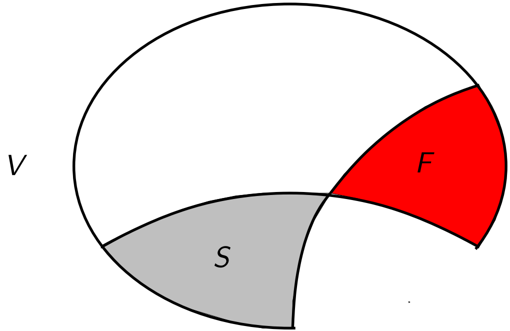
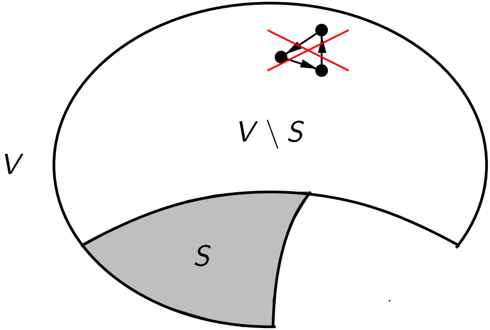


Vertices from $X \setminus S$ can be deleted immediately.

$\Rightarrow$ New task:
Given a graph $G = (V, E)$ and a solution S of size $k + 1$, compute a minimum solution F with $F \cap S = \emptyset$.

Compression Routine (3)



Observations:

- $G[V \setminus S]$ is acyclic.

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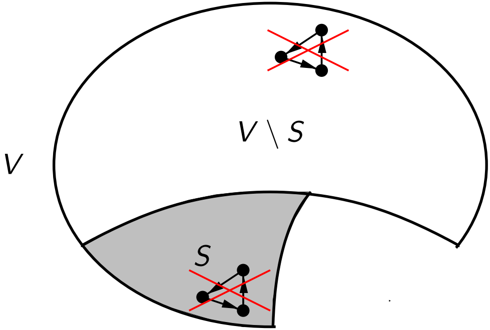

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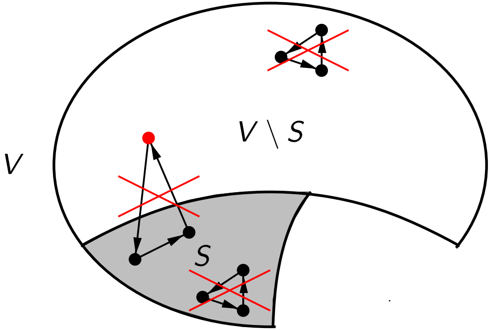
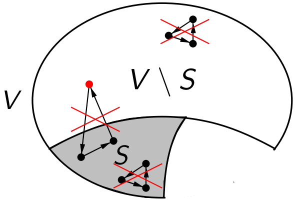


Observations:

- $G[V \setminus S]$ is acyclic.
- $G[S]$ must be acyclic.
- All triangles with exactly one vertex in $V \setminus S$ can easily be destroyed.

Topological Sorts

- $G[S]$ is acyclic $\Rightarrow$
  S has topological sort $s_1, \ldots, s_{|S|}$.
- $G[V \setminus S]$ is acyclic $\Rightarrow$
  $V \setminus S$ has topological sort.



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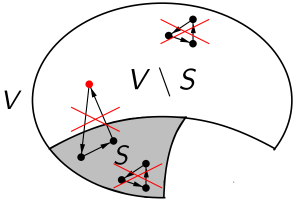
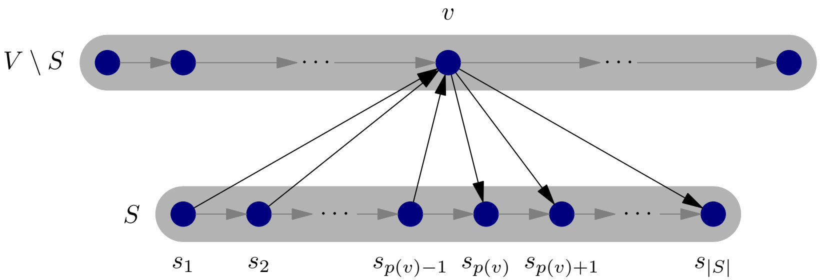
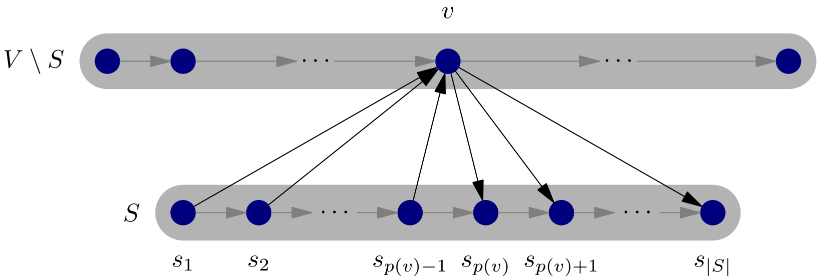




Goal: Insert a maximum subset of $V \setminus S$ into $s_1, \ldots, s_{|S|}$.

Observation: Every vertex v of $V \setminus S$ has a "natural position" $p(v)$ relative to the vertices of S.

The "Natural Position" $p(v)$



Definition of $p(v)$:
For all $s_i \in S$ with $i < p(v)$: edge from s_i to v.
For all $s_i \in S$ with $i \geq p(v)$: edge from v to s_i.

The value $p(v)$ is defined and unique for every $v \in V \setminus S$.

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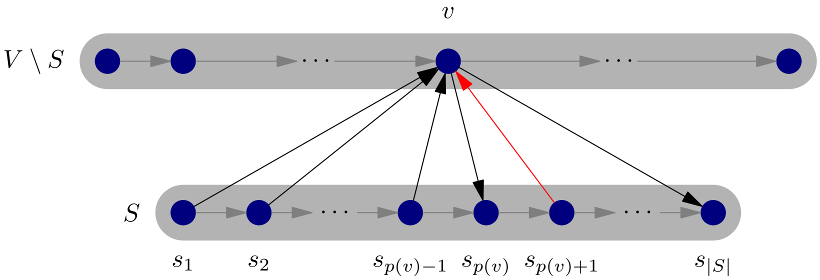

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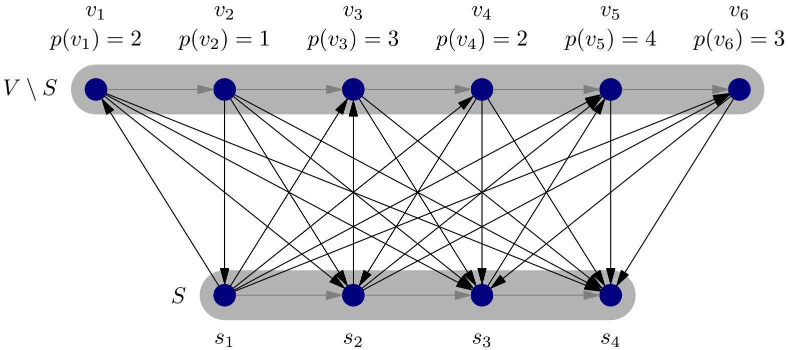
The Goal Graph



The vertices $(V \setminus S) \setminus F$ must be sortable in such a way that

- the topological sort of $V \setminus S$ is preserved, and
- the sort of $V \setminus S$ by "natural position" p is preserved.

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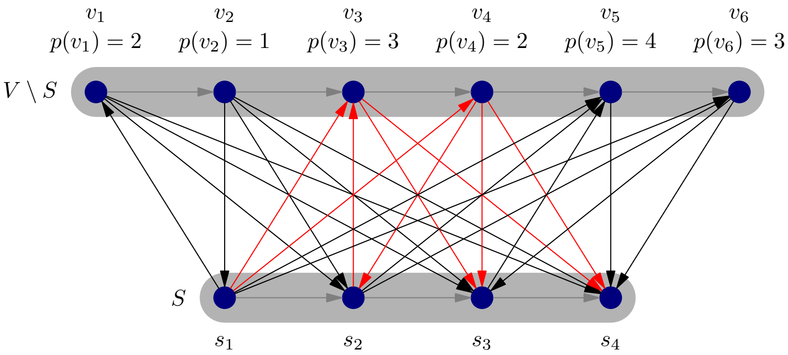

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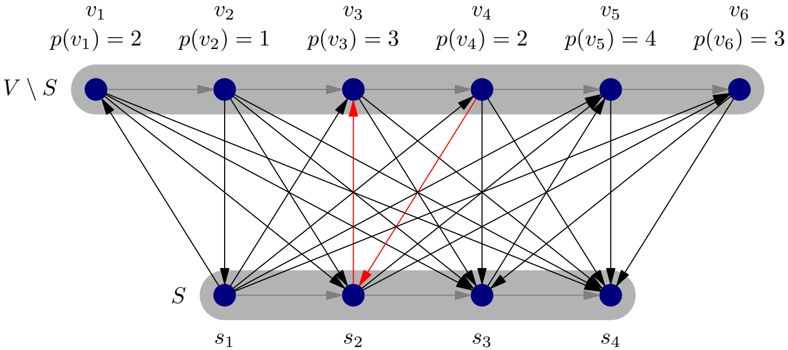

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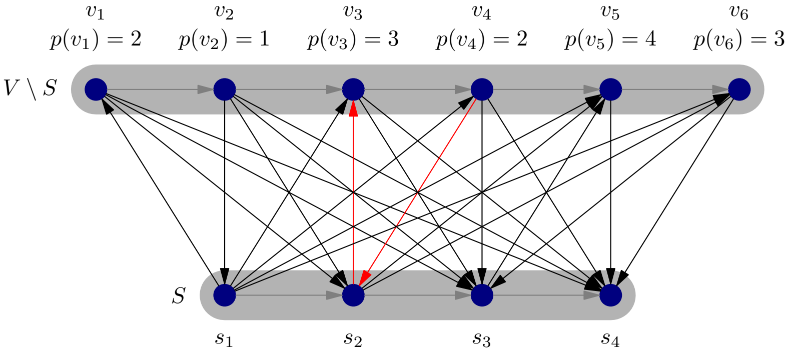

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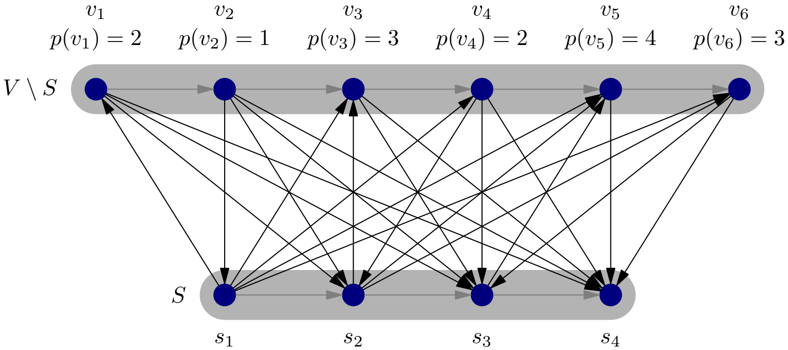
$\Rightarrow$ Search for the longest common subsequence of both sorts.

Example

- $V \setminus S$ sorted topologically: $v_1 v_2 v_3 v_4 v_5 v_6$
- $V \setminus S$ sorted by p: $v_2 v_1 v_4 v_3 v_6 v_5$
- A longest common subsequence is $v_1 v_3 v_5$



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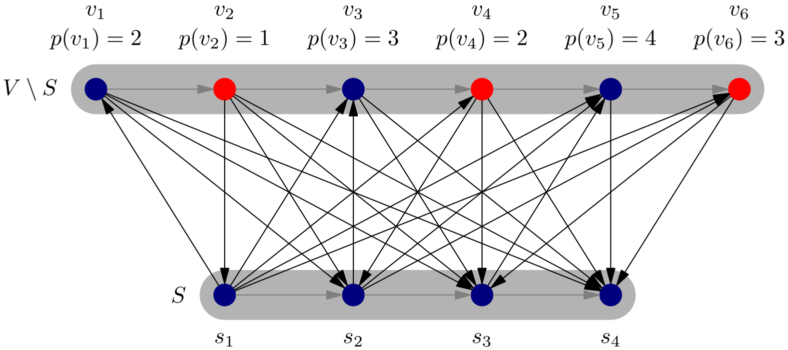

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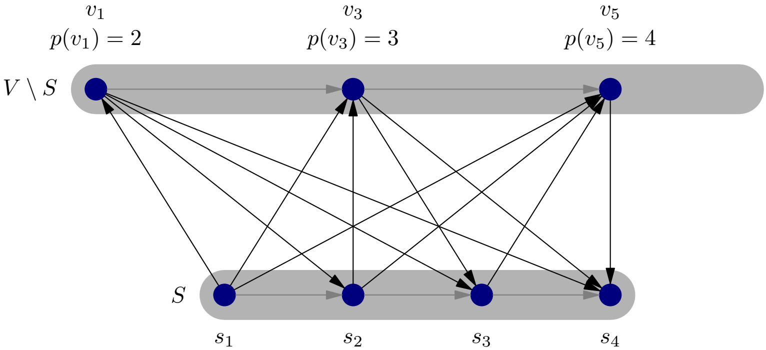

Overall Running Time

Time consumption:

- n iterations of the compression routine;
- 2^{k+1} partitions per iteration;
- time $O(n \cdot k)$ for destroying triangles, time $O(n \log n)$ for sorting vertices and finding the longest common subsequence.

Running time for solving Feedback Vertex Set in tournaments:

$$O(2^k \cdot n^2(\log n + k))$$

Further Results

- Fixed-parameter search tree algorithm for Feedback Vertex Set in bipartite tournaments.
- Problem kernel for Feedback Vertex Set in tournaments.
- Problem kernel for Feedback Arc Set in tournaments.

Outlook

- The parameterized complexity of FEEDBACK VERTEX SET on directed graphs is a currently open problem.
- Feedback set problems in bipartite tournaments are not well explored:
  - Problem kernel for FEEDBACK VERTEX SET.
  - Complexity of FEEDBACK ARC SET.
  - Problem kernel for FEEDBACK ARC SET.
- Implementation of our algorithm, experiments.