The Search for Consecutive Ones Submatrices: Faster and More General

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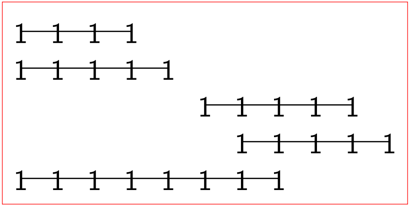
Algorithms and Complexity in Durham 2007

Consecutive Ones Property (C1P)



A 0/1-matrix has the C1P if its columns can be permuted such that in each row the 1's form a block.

Consecutive Ones Property (C1P)

Example for a matrix having the C1P:

1	2	3	4	5
1	1			1
1		1		1
1		1	1	

Consecutive Ones Property (C1P)

Example for a matrix having the C1P:

1	2	3	4	5
1	1			1
1		1		1
1		1	1	

2	5	1	3	4
1	1	1		
	1	1	1	
		1	1	1

Consecutive Ones Property (C1P)

Examples for matrices **not** having the C1P:

1	1	0
0	1	1
1	0	1

1	1	0	0
0	1	1	0
0	0	1	1
1	0	0	1

1	1	0	0	0
0	1	1	0	0
0	0	1	1	0
0	0	0	1	1
1	0	0	0	1

1	1	0	0
0	1	1	0
0	1	0	1

1	1	0	0	0	0
0	0	1	1	0	0
0	0	0	0	1	1
1	0	1	0	1	0

Consecutive Ones Property (C1P)

The Consecutive Ones Property...

- ...expresses "locality" of the input data.
- ...appears in many applications, e.g.
  - in railway system optimization
    [Ruf, Schöbel, Discrete Optimization, 2004;
    Mecke, Wagner, ESA '04],
  - bioinformatics
    [Christof, Oswald, Reinelt, IPCO '98;
    Lu, Hsu, J. Comp. Biology, 2003].
- ...can be recognized in polynomial time
  [Booth, Lueker, J. Comput. System Sci., 1976;
  Meidanis, Porto, Telles, Discrete Appl. Math., 1998;
  Habib, McConnell, Paul, Viennot, Theor. Comput. Sci., 2000,
  Hsu, J. Algorithms, 2002; McConnell, SODA '04].
- ...is subject of current research
  [Hajiaghayi, Ganjali, Inf. Process. Lett., 2002;
  Tan, Zhang, Algorithmica, 2007].

Problem Definition

Min-COS-C (Min-COS-R)
Given: A matrix M and a positive integer k.
Question: Can we delete at most k columns (at most k rows) such that the resulting matrix has the C1P?

Known and New Results

Min-COS-C:

- NP-hard for $(2,3)$- and $(3,2)$-matrices[1]
- Approximation algorithms for maximization version on $(2,3)$-, $(3,2)$-, and $(2,*)$-matrices[1]
- FPT and approximation results for $(*,2)$-, $(2,*)$- and $(*,\Delta)$-matrices[2]

Min-COS-R:

- NP-hard for $(3,2)$-matrices[3]

[1] [Tan, Zhang, ISAAC '04]
[2] [Dom, Guo, Niedermeier, TAMC '07]
[3] [Garey, Johnson, 1979; Hajiaghayi, Ganjali, Inf. Process. Lett., 2002]

Known and New Results

Min-COS-C:

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- FPT and approximation results for $(*,2)$-, $(2,*)$- and $(*,\Delta)$-matrices[2]
- Improved results for $(*,\Delta)$-matrices (FPT w.r.t (k,Δ))

Min-COS-R:

- NP-hard for $(3,2)$-matrices[3]
- FPT and approximation results for $(*,\Delta)$-matrices

[1] [Tan, Zhang, ISAAC '04]
[2] [Dom, Guo, Niedermeier, TAMC '07]
[3] [Garey, Johnson, 1979; Hajiaghayi, Ganjali, Inf. Process. Lett., 2002]

Structure of What Follows

- **Algorithmic Framework**
- From Circ1P to C1P

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

$$M_{\mathrm{I}_p},\ p \geq 1: \quad \underbrace{\left.\begin{array}{cccccc} \mathbf{1} & \mathbf{1} & 0 & \cdots & & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & \cdots & 0 \\ & & & \cdots & & \\ 0 & \cdots & & 0 & \mathbf{1} & \mathbf{1} \\ \hline \mathbf{1} & 0 & \cdots & & 0 & \mathbf{1} \end{array}\right\} p+2}_{p+2}$$

$$M_{\mathrm{II}_p},\ p \geq 1: \quad \underbrace{\left.\begin{array}{cccccc|c} \mathbf{1} & \mathbf{1} & 0 & & \cdots & & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & \cdots & & 0 \\ & & & \cdots & & & \\ 0 & \cdots & & 0 & \mathbf{1} & \mathbf{1} & 0 \\ \hline 0 & \mathbf{1} & & \cdots & & & \mathbf{1} \\ \mathbf{1} & & \cdots & & \mathbf{1} & 0 & \mathbf{1} \end{array}\right\} p+3}_{p+3}$$

$$M_{\mathrm{III}_p},\ p \geq 1: \quad \underbrace{\left.\begin{array}{cccccc|c} \mathbf{1} & \mathbf{1} & 0 & & \cdots & & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & \cdots & & 0 \\ & & & \cdots & & & \\ 0 & \cdots & & 0 & \mathbf{1} & \mathbf{1} & 0 \\ \hline 0 & \mathbf{1} & \cdots & & \mathbf{1} & 0 & \mathbf{1} \end{array}\right\} p+2}_{p+3}$$

$$M_{\mathrm{IV}}: \quad \begin{array}{cccccc} \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \end{array} \qquad M_{\mathrm{V}}: \quad \begin{array}{ccccc} \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} \end{array}$$

Theorem: A matrix has the C1P iff it contains none of the shown matrices.

[Tucker, Journal of Combinatorial Theory (B), 1972]

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

$$M_{\mathrm{I}_p} = \overbrace{\left.\begin{array}{|ccccc|}\hline \mathbf{1} & \mathbf{1} & 0 & \cdots & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 \cdots & 0 \\ & & \cdots & & \\ 0 & \cdots & 0 & \mathbf{1} & \mathbf{1} \\ \hline \mathbf{1} & 0 & \cdots & 0 & \mathbf{1} \\ \hline \end{array}\right\}}^{p+2} p+2, \quad p \geq 1$$

$$M_{\mathrm{II}_p} = \overbrace{\left.\begin{array}{|cccccc|c|}\hline \mathbf{1} & \mathbf{1} & 0 & & \cdots & & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & \cdots & & 0 \\ & & & \cdots & & & \\ 0 & & \cdots & 0 & \mathbf{1} & \mathbf{1} & 0 \\ \hline 0 & \mathbf{1} & & & \cdots & & \mathbf{1} \\ \mathbf{1} & & \cdots & & \mathbf{1} & 0 & \mathbf{1} \\ \hline \end{array}\right\}}^{p+3} p+3, \quad p \geq 1$$

$$M_{\mathrm{III}_p} = \overbrace{\left.\begin{array}{|cccccc|c|}\hline \mathbf{1} & \mathbf{1} & 0 & & \cdots & & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & \cdots & & 0 \\ & & & \cdots & & & \\ 0 & & \cdots & 0 & \mathbf{1} & \mathbf{1} & 0 \\ \hline 0 & \mathbf{1} & & \cdots & \mathbf{1} & 0 & \mathbf{1} \\ \hline \end{array}\right\}}^{p+3} p+2, \quad p \geq 1$$

$$M_{\mathrm{IV}} = \begin{array}{|cccccc|}\hline \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\ \hline \end{array} \qquad M_{\mathrm{V}} = \begin{array}{|ccccc|}\hline \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} \\ \hline \end{array}$$

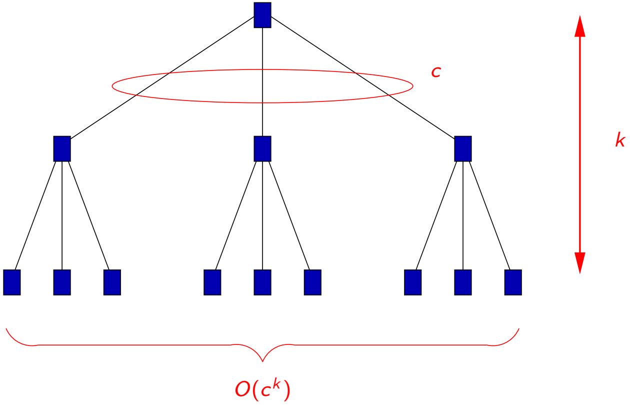
Approach: Use a search tree algorithm.

Repeat:

1. Search for a "forbidden submatrix".
2. Branch on which of its columns has to be deleted.

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

Search Tree Algorithm:



Finite size c of forbidden matrices $\Rightarrow$ search tree of size $O(c^k)$.
(Alternatively: Factor-c approximation algorithm.)

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

$$M_{\mathrm{I}_p},\ p \geq 1: \quad \overbrace{\left.\begin{array}{|ccccc|} \hline 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 \cdots & 0 \\ & & \cdots & & \\ 0 & \cdots & 0 & 1 & 1 \\ \hline 1 & 0 & \cdots & 0 & 1 \\ \hline \end{array}\right\}}^{p+2} p+2$$

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$$M_{\mathrm{III}_p},\ p \geq 1: \quad \overbrace{\left.\begin{array}{|cccccc|c|} \hline 1 & 1 & 0 & & \cdots & & 0 \\ 0 & 1 & 1 & 0 & \cdots & & 0 \\ & & & \cdots & & & \\ 0 & \cdots & & 0 & 1 & 1 & 0 \\ \hline 0 & 1 & \cdots & & 1 & 0 & 1 \\ \hline \end{array}\right\}}^{p+3} p+2$$

$$M_{\mathrm{IV}}: \begin{array}{|cccccc|} \hline 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ \hline \end{array} \qquad M_{\mathrm{V}}: \begin{array}{|ccccc|} \hline 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ \hline \end{array}$$

A $(*, \Delta)$-matrix can contain

- M_{I_p} with unbounded size,
- M_{II_p} with $1 \leq p \leq \Delta - 2$,
- M_{III_p} with $1 \leq p \leq \Delta - 1$,
- M_{IV}, and M_{V}.

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

Problem: Matrices M_{I_p} of unbounded size can occur.

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

Problem: Matrices M_{I_p} of unbounded size can occur.

Idea: First destroy all "small" forbidden submatrices (search tree algorithm), and then see what happens. . .

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

Algorithmic framework for Min-COS-C / Min-COS-R:

1. Destroy the submatrices from

$$X := \{M_{\mathrm{I}_p} \mid 1 \leq p \leq \Delta - 1\} \cup \{M_{\mathrm{II}_p} \mid 1 \leq p \leq \Delta - 2\} \\ \cup \{M_{\mathrm{III}_p} \mid 1 \leq p \leq \Delta - 1\} \cup \{M_{\mathrm{IV}}, M_{\mathrm{V}}\}.$$

2. Destroy the remaining M_{I_p} $(p \geq \Delta)$.

Theorem: If a $(*, \Delta)$-matrix M contains none of the matrices in X as a submatrix, then M can be partitioned into "independent" submatrices that have the *"circular ones property (Circ1P)"*.

[Dom, Guo, Niedermeier, TAMC '07]

Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

FPT algorithm:

Running time:

	$(\|\text{submatrix}\|)^k$	$\cdot$	(search	$+$	"Circ1P$\rightarrow$C1P" time)	
Old[4]**:**	$(\Delta+2)^k$	$\cdot$	$(n^{O(\Delta)}$	$+$	$n^{O(\Delta)})$	(only Min-COS-C)
New:	$(\Delta+2)^k$	$\cdot$	$(n^{O(1)}$	$+$	$O(\Delta mn))$	

Approximation algorithm:

Approximation factor: $|\text{submatrix}|$

Running time: $k \cdot$ (search + "Circ1P$\rightarrow$ C1P" time)

[4] [Dom, Guo, Niedermeier, TAMC '07]

Structure of the Talk

- Algorithmic Framework
- **From Circ1P to C1P**

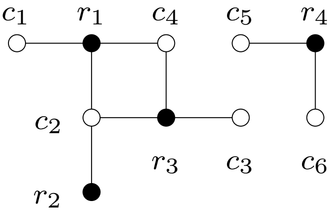
From Circ1P to C1P

Again:

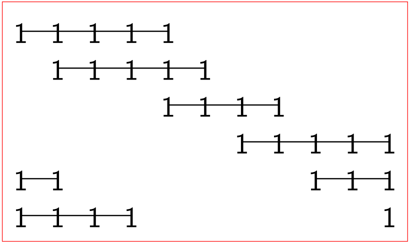
Theorem: If a $(*, \Delta)$-matrix M contains none of the matrices in X as a submatrix, then M can be partitioned into "independent" submatrices that have the *"circular ones property (Circ1P)"*.
[Dom, Guo, Niedermeier, TAMC '07]

"Independent" Submatrices

	c_1	c_2	c_3	c_4	c_5	c_6
r_1	**1**	**1**	0	**1**	0	0
r_2	0	**1**	0	0	0	0
r_3	0	**1**	**1**	**1**	0	0
r_4	0	0	0	0	**1**	**1**



The Circular Ones Property (Circ1P)



A 0/1-matrix M has the Circ1P if its columns can be permuted such that in each row the 1's form a block *when M is wrapped around a vertical cylinder*.

From Circ1P to C1P

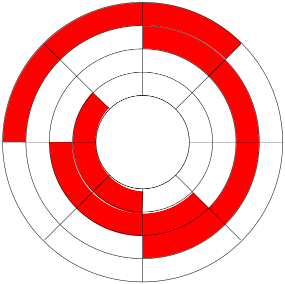
C1P:	1's blockwise after column permutations
Circ1P:	1's blockwise on a cylinder after column permutations
strong C1P:	1's blockwise *without* column permutations
strong Circ1P:	1's blockwise on a cylinder *without* column permutations

(Circ1P/C1P means: Strong Circ1P/strong C1P can be obtained by column permutations.)

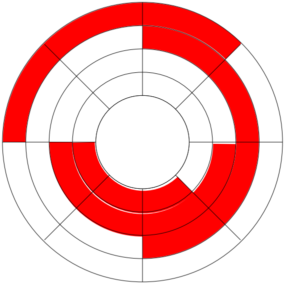
From Circ1P to C1P

We imagine the matrices as wrapped around a vertical cylinder.

Strong Circ1P:



Strong C1P:



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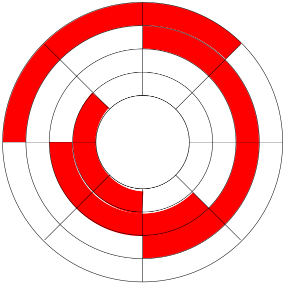
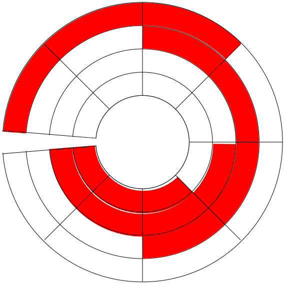


Strong C1P:



Strong C1P =
Strong Circ1P + "cut"

From Circ1P to C1P

Our task:

strong Circ1P	$\longrightarrow$ column deletions	strong Circ1P + C1P

From Circ1P to C1P

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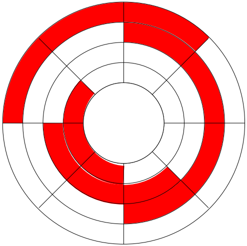
First consider this task:

strong Circ1P	$\longrightarrow$ column deletions	strong Circ1P + strong C1P



From Circ1P to C1P

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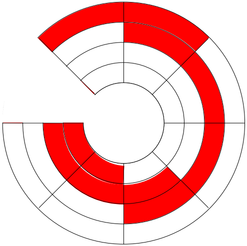
First consider this task:

strong Circ1P	$\longrightarrow$ column deletions	strong Circ1P + strong C1P



Obs.: Deleting a consecutive set of columns is always optimal.

From Circ1P to C1P

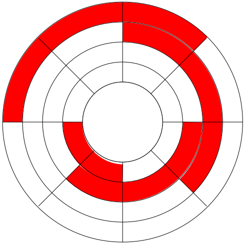
Our task:

strong Circ1P	$\longrightarrow$ column deletions	strong Circ1P + C1P

Easy task:

strong Circ1P	$\longrightarrow$ column deletions	strong Circ1P + strong C1P

We hope: Does "strong Circ1P + C1P" imply "strong C1P"?



From Circ1P to C1P

Conjecture: If a matrix has the strong Circ1P and the C1P, then it has also the strong C1P.

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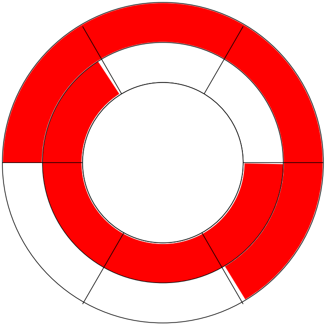

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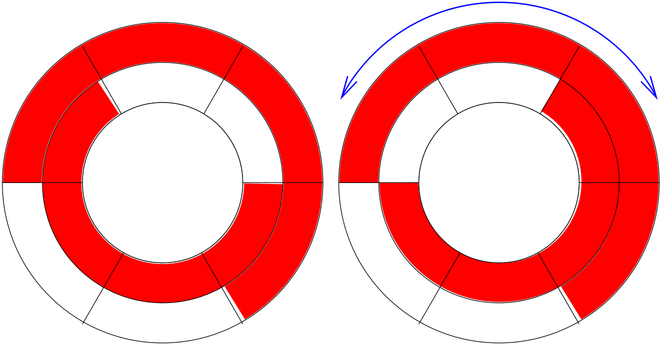

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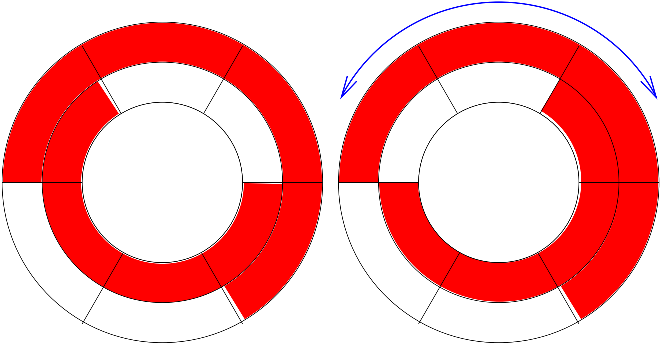

New conjecture: If a matrix with $\geq 2\Delta - 1$ columns has the strong Circ1P and the C1P, then it has also the strong C1P.

From Circ1P to C1P

To be proven: If a matrix with $\geq 2\Delta - 1$ columns has the strong Circ1P and the C1P, then it has also the strong C1P.

Very helpful:
Theorem: Let M have the strong Circ1P. Then *every* column permutation that also yields the strong Circ1P can be obtained by a series of circular module reversals.
[Hsu, McConnell, Theor. Comput. Sci., 2003]

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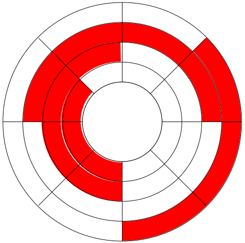

strong Circ1P+ C1P

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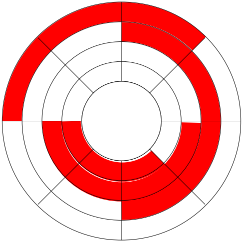
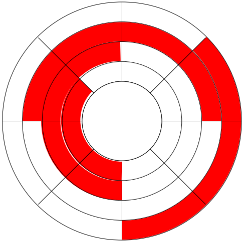


strong Circ1P+ strong C1P



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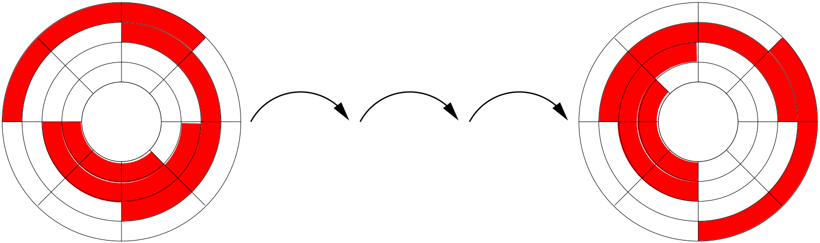

strong Circ1P+ strong C1P

strong Circ1P+ C1P

From Circ1P to C1P

Now to be proven: Let M be a matrix with with $\geq 2\Delta - 1$ columns that has the strong Circ1P and the strong C1P. Reversing an arbitrary circular module of M does not affect these properties.

From Circ1P to C1P

Algorithm for Min-COS-C on matrices with Circ1P:

1. Permute the columns to get the strong Circ1P.
2. Search for a set of *consecutive* consecutive columns whose deletion yields the strong C1P.

Main Open Question

How can a matrix that has the (strong) Circ1P be modified by deleting a minimum number of 1-entries such that the resulting matrix has the C1P?